出版前言

2011 年 12 月 11 日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了"纪念钱学森诞辰 100 周年"系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共 10 卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

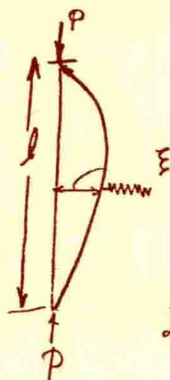
Contents

Section 1

Ring Supported Column

SHELL CALCULATIONS

Ring Supported Column



$$w = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

The bending energy

$$\frac{1}{2} EI \int_0^l \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = \frac{1}{2} EI \int_0^l \left\{\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}\right\}^2 dx$$

$$= \frac{EIl}{4} \sum_{n=1}^{\infty} a_n^2 \left(\frac{n\pi}{l}\right)^4$$

The decrease in potential of P

$$\frac{1}{2} P \int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx = \frac{1}{4} Pl \sum_{n=1}^{\infty} a_n^2 \left(\frac{n\pi}{l}\right)^2$$

The strain energy of spring $\quad S(\xi)$

$$\xi = a_1 - a_3 + a_5 - a_7 + a_9 - \cdots$$

$$\therefore \quad \frac{EIl}{4} \frac{\pi^4}{l^4} \sum_{n=1}^{\infty} n^4 a_n^2 + S(\xi) - \frac{Pl}{4} \frac{\pi^2}{l^2} \sum_{n=1}^{\infty} n^2 a_n^2 = \mathcal{E}$$

for the antisymmetric coefficient, we have

$$P = \frac{n^2 \pi^2 EI}{l^2}$$

for the symmetric coefficient, n odd,

$$a_n \frac{EIl}{2} \frac{\pi^4}{l^4} n^4 + (-1)^{\frac{n-1}{2}} F(\xi) = \frac{Pl}{2} \frac{\pi^2}{l^2} n^2 a_n$$

$$a_n \frac{l}{2} \frac{\pi^2}{l^2} n^2 \left[EI \frac{\pi^2}{l^2} n^2 - P \right] = (-1)^{\frac{n+1}{2}} F(\xi)$$

$$a_n = \frac{(-1)^{\frac{n+1}{2}} F(\xi)}{\frac{l}{2} \frac{\pi^2}{l^2} n^2 \left[EI \frac{\pi^2}{l^2} n^2 - P \right]}$$

$$\xi = -\sum_{n=1,3,5}^{\infty} \frac{F(\xi)}{\frac{l}{2} \frac{\pi^2}{l^2} n^2 \left[EI \frac{\pi^2}{l^2} n^2 - P \right]}$$

$$\frac{\xi}{F(\xi)} = -\sum_{n=1,3,5}^{\infty} \frac{1}{\frac{l}{2} \frac{\pi^2}{l^2} n^2 \left[EI \frac{\pi^2}{l^2} n^2 - P \right]}$$

$$= \sum_{n=1,3,5}^{\infty} \frac{1}{\frac{l}{2} \frac{\pi^2}{l^2} n^2 \frac{EI\pi^2}{l^2} \left[\frac{P}{P_{Eu_1}} - n^2 \right]}$$

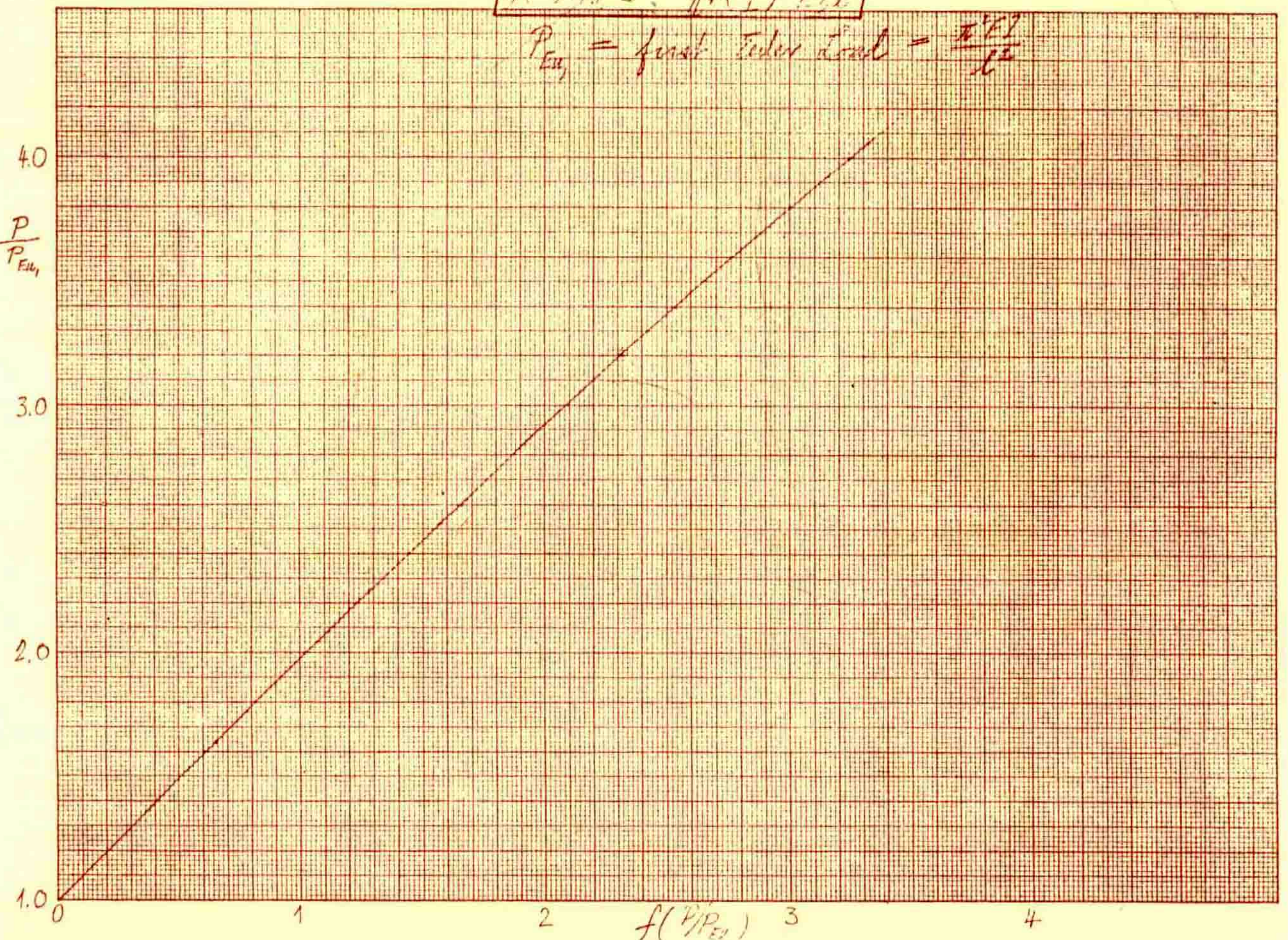
$$\frac{l}{2}\frac{\pi^2}{l^2}\frac{EI\pi^2}{l^2}\frac{\xi}{F(\xi)} = \sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left[\frac{P}{P_{Eu_1}}-n^2\right]}$$

$$\boxed{\frac{1}{2}\left(\frac{\pi}{l}\right)^4 EI\,\frac{\xi l}{F(\xi)} = \sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left[\frac{P}{P_{E1}}-n^2\right]}}$$

① P/P_{Eu_1}	② $\sum_{n=1,3,5}^{\infty}\frac{1}{n^2[P/P_E - n^2]}$	③ $1/②$
4.0	0.3086	3.240
3.8	0.3333	3.000
3.6	0.3616	2.765
3.4	0.3944	2.535
3.2	0.4330	2.309
3.0	0.4791	2.087
2.8	0.5352	1.868
2.6	0.6053	1.652
2.4	0.6951	1.439
2.2	0.8146	1.228
2.0	0.9818	1.0185
1.8	1.2323	0.8115
1.6	1.6494	0.6063
1.4	2.4831	0.4027
1.2	4.9835	0.2007
1.1	9.9837	0.1002

P/P_{Eu}	$n = 1$		$n = 3$		$n = 5$		$n = 7$		$n = 9$	
	$n^2(P/P_E - n^2)$	$1/(\quad)$	$n^2(P/P_E - n^2)$	$1/(\quad)$	$n^2(P/P_{Eu} - n^2)$	$1/(\quad)$	$n^2(P/P_E - n^2)$	$1/(\quad)$	$n^2(P/P_E - n^2)$	$1/(\quad)$
4.0	3	0.33333	-45	-0.02222	-525	-0.00191	-2205	-0.00045	-6237.0	-0.00016
3.8	2.8	0.35714	-46.8	-0.02137	-530	-0.00189	-2214.8	-0.00045	-6253.2	"
3.6	2.6	0.38462	-48.6	-0.02058	-535	-0.00187	-2224.6	-0.00045	-6269.4	"
3.4	2.4	0.41667	-50.4	-0.01984	-540	-0.00185	-2234.4	-0.00045	-6285.6	"
3.2	2.2	0.45455	-52.2	-0.01916	-545	-0.00183	-2244.2	-0.00045	-6301.8	"
3.0	2.0	0.50000	-54.0	-0.01852	-550	-0.00182	-2254.0	-0.00044	-6318.0	"
2.8	1.8	0.55556	-55.8	-0.01792	-555	-0.00180	-2263.8	"	-6334.2	"
2.6	1.6	0.62500	-57.6	-0.01736	-560	-0.00178	-2273.6	"	-6350.4	"
2.4	1.4	0.71429	-59.4	-0.01684	-565	-0.00177	-2283.4	"	-6366.6	"
2.2	1.2	0.83333	-61.2	-0.01634	-570	-0.00175	-2293.2	"	-6382.8	"
2.0	1.0	1.00000	-63.0	-0.01587	-575	-0.00174	-2303.0	-0.00041	-6399.0	"
1.8	0.8	1.25000	-64.8	-0.01543	-580	-0.00172	-2312.8	"	-6415.2	"
1.6	0.6	1.66667	-66.6	-0.01502	-585	-0.00171	-2322.6	"	-6431.4	"
1.4	0.4	2.5000	-68.4	-0.01462	-590	-0.00169	-2332.4	"	-6447.6	"
1.2	0.2	5.0000	-70.2	-0.01425	-595	-0.00168	-2342.2	"	-6463.8	-0.00015
1.1	0.1	10.0000	-71.1	-0.01406	-597.5	-0.00167	-2347.1	"	-6471.9	"

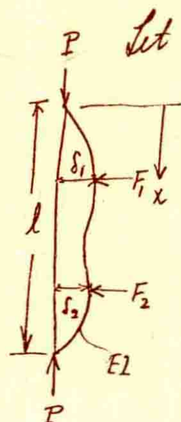
KEUFFEL & ESSER CO., N. Y. NO. 359-14
Millimeters, 10th lines heavy.
MADE IN U. S. A.



Section 2

Buckling of Column with Two Non-linear Suppertes

Buckling of Column with two non-linear supports



Let $w = \sum_{n=1,2,3}^{\infty} a_n \sin \frac{n\pi x}{l}$

The lowering of the potential of P

$$= -\frac{1}{2} P \int_0^l \left(\frac{dw}{dx}\right)^2 dx$$

$$= -\frac{1}{2} P \frac{l}{2} \sum_{n=1,2,3}^{\infty} \left(\frac{n\pi}{l}\right)^2 a_n^2$$

The increase in bending strain energy

$$= \frac{EI}{2} \int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2} \frac{l}{2} \sum_{n=1,2,3}^{\infty} \left(\frac{n\pi}{l}\right)^4 a_n^2$$

W_1 = work done on F_1

W_2 = " " on F_2

Total potential of the system

$$= \frac{l}{4}\left(\frac{\pi}{l}\right)^2 \left\{ \sum_{n=1,2,3}^{\infty} n^2 \left[EI\left(\frac{n\pi}{l}\right)^2 - P\right] a_n^2 \right\} + W_1 + W_2$$

The equilibrium condition is

$$\frac{l}{2}\left(\frac{\pi}{l}\right)^2 n^2 \left[EI\left(\frac{n\pi}{l}\right)^2 - P \right] a_n + \sin\frac{n\pi}{3} F_1 + \sin\frac{2n\pi}{3} F_2 = 0.$$

$$a_n = \frac{2}{l n^2 \left(\frac{\pi}{l}\right)^2} \frac{\sin\frac{n\pi}{3} F_1 + \sin\frac{2n\pi}{3} F_2}{\left[-P - EI\left(\frac{n\pi}{l}\right)^2 \right]}$$

$$P_E = \frac{\pi^2 EI}{l^2}$$

$$= \frac{2}{n^2 \frac{\pi^2}{l}} \frac{\sin\frac{n\pi}{3} F_1 + \sin\frac{2n\pi}{3} F_2}{\left[P - P_E n^2 \right]}$$

$$\boxed{\frac{a_n}{l} = \frac{2}{\pi^2} \frac{\sin\frac{n\pi}{3}\left(\frac{F_1}{P_E}\right) + \sin\frac{2n\pi}{3}\left(\frac{F_2}{P_E}\right)}{n^2\left[\frac{P}{P_E} - n^2\right]}}$$

$$\left.\begin{aligned} \frac{\delta_1}{l} &= \frac{2}{\pi^2} \sum_{n=1,2,3}^{\infty} \frac{\sin^2\frac{n\pi}{3}\left(\frac{F_1}{P_E}\right) + \sin\frac{n\pi}{3}\sin\frac{2n\pi}{3}\left(\frac{F_2}{F_E}\right)}{n^2\left[\frac{P}{P_E} - n^2\right]} \\ \frac{\delta_2}{l} &= \frac{2}{\pi^2} \sum_{n=1,2,3}^{\infty} \frac{\sin\frac{n\pi}{3}\sin\frac{2n\pi}{3}\left(\frac{F_1}{P_E}\right) + \sin^2\frac{2n\pi}{3}\left(\frac{F_2}{F_E}\right)}{n^2\left[\frac{P}{P_E} - n^2\right]} \end{aligned}\right\}$$

$$A = \frac{2}{\pi^2} \sum_{n=1,2,3}^{\infty} \frac{\sin^2 \frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]} = \frac{1}{\pi^2} \sum_{n=1,2,3}^{\infty} \frac{1 - \cos\frac{2n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

$$\boxed{A = \frac{1}{\pi^2 \frac{P}{P_E}} \sum_{n=1,2,3}^{\infty} \left\{1 - \cos\frac{2n\pi}{3}\right\} \left\{ \frac{1}{n^2} + \frac{1}{\frac{P}{P_E} - n^2} \right\}}$$

$$\boxed{\sum_{n=1,2,3}^{\infty} \frac{1}{n^2} = \zeta(2) = 2\,\frac{\pi^2}{2}\,\frac{1}{6} = \frac{\pi^2}{6}}$$

$$\sum_{n=1,2,3}^{\infty} \frac{1}{\frac{P}{P_E} - n^2} = \frac{1}{2\sqrt{\frac{P}{P_E}}} \sum_{n=1,2,3}^{\infty} \left\{ \frac{1}{\sqrt{\frac{P}{P_E}} - n} + \frac{1}{\sqrt{\frac{P}{P_E}} + n} \right\}$$

$$\sum_{n=1,2,3}^{\infty} \frac{1}{\frac{P}{P_E} - n^2} = \sum_{n=1,2,3} \frac{1}{\frac{P}{P_E} - n^2} - \sum_{n=4,5,6}^{\infty} \frac{1}{n^2 - \frac{P}{P_E}}$$

$$= \sum_{n=1}^{3} \frac{1}{\frac{P}{P_E} - n^2} - \frac{1}{2\sqrt{\frac{P}{P_E}}} \sum_{n=4}^{\infty} \left\{ \frac{1}{n - \sqrt{\frac{P}{P_E}}} - \frac{1}{n + \sqrt{\frac{P}{P_E}}} \right\}$$

$$= \sum_{n=1}^{3} \frac{1}{\frac{P}{P_E} - n^2} - \frac{1}{2\sqrt{\frac{P}{P_E}}} \sum_{n=4}^{\infty} \left\{ \int_0^{\infty} e^{-x\left(n - \sqrt{\frac{P}{P_E}}\right)} dx - \int_0^{\infty} e^{-x\left(n + \sqrt{\frac{P}{P_E}}\right)} dx \right\}$$

$$\sum_{n=4}^{\infty}\int_0^{\infty} e^{-x\left(n-\sqrt{\frac{p}{p_E}}\right)}dx = \int_0^{\infty} e^{+x\sqrt{\frac{p}{p_E}}}\sum_{n=4}^{\infty}\left(e^{-x}\right)^n dx$$

$$= \int_0^{\infty} e^{x\sqrt{\frac{p}{p_E}}}\, e^{-4x}\sum_{n=0}^{\infty}\left(e^{-x}\right)^n dx = \int_0^{\infty} e^{x\sqrt{\frac{p}{p_E}}}\frac{e^{-4x}}{1-e^{-x}}dx$$

$$= \int_0^{\infty}\frac{e^{-x\left(4-\sqrt{\frac{p}{p_E}}\right)}}{1-e^{-x}}dx$$

$$\sum_{n=4}^{\infty}\int_0^{\infty} e^{-x\left(n+\sqrt{\frac{p}{p_E}}\right)}dx = \int_0^{\infty}\frac{e^{-x\left(4+\sqrt{\frac{p}{p_E}}\right)}}{1-e^{-x}}dx$$

$$\therefore \boxed{\sum_{n=1,2,3}^{\infty}\frac{1}{\frac{p}{p_E}-n^2} = \sum_{n=1}^{3}\frac{1}{\frac{p}{p_E}-n^2} - \frac{1}{2\sqrt{\frac{p}{p_E}}}\left\{\psi\left(4+\sqrt{\frac{p}{p_E}}\right)-\psi\left(4-\sqrt{\frac{p}{p_E}}\right)\right\}}$$

$$\sum_{n=1,2,3}^{\infty}\frac{\cos\frac{2n\pi}{3}}{n^2} = \sum_{n=1,3,5}^{\infty}\frac{\cos\frac{2n\pi}{3}}{n^2} + \frac{1}{4}\sum_{n=1,2,3}^{\infty}\frac{\cos\frac{4n\pi}{3}}{n^2}$$

However $\cos\frac{4n\pi}{3} = \cos n\left(\frac{4\pi}{3}\right) = \cos n\left(2\pi-\frac{2\pi}{3}\right) = \cos\frac{2n\pi}{3}$

$$\therefore \boxed{\begin{aligned}\sum_{n=1,2,3}^{\infty}\frac{\cos\frac{2n\pi}{3}}{n^2} &= \frac{4}{3}\sum_{n=1,3,5}^{\infty}\frac{\cos\frac{2n\pi}{3}}{n^2} = \frac{\pi}{4}\frac{4}{3}\,\phi\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)\\ &= \frac{\pi}{3}\,\phi\left(\frac{7\pi}{6}\right) = -\frac{\pi}{3}\cdot\frac{1}{6}\pi = -\frac{\pi^2}{18}\end{aligned}}$$

See K.B, p.39

$$\frac{\delta_1}{l} = A\left(\frac{F_1}{P_E}\right) + B\left(\frac{F_2}{P_E}\right)$$

$$\frac{\delta_2}{l} = B\left(\frac{F_1}{P_E}\right) + A\left(\frac{F_2}{P_E}\right)$$

$$\sin\frac{2n\pi}{3} = \sin n\left(\frac{2\pi}{3}\right) = \sin n\left(\pi - \frac{\pi}{3}\right) = -(-)^n \sin\frac{n\pi}{3}$$

where

$$A = \frac{2}{\pi^2}\sum_{n=1,2,3}^{\infty} \frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

$$B = -\frac{2}{\pi^2}\sum_{n=1,2,3}^{\infty} (-)^n \frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

$$A = A\left(\frac{P}{P_E}\right)$$

$$B = B\left(\frac{P}{P_E}\right)$$

$$\frac{a_n}{l} = \frac{2}{\pi^2}\,\frac{\sin\frac{n\pi}{3}\left[\left(\frac{F_1}{P_E}\right) - (-1)^n\left(\frac{F_2}{P_E}\right)\right]}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

$$\frac{\mathcal{E}}{l} = \frac{1}{4}\sum_{n=1,2,3}^{\infty} (n\pi)^2\left(\frac{a_n}{l}\right)^2$$

$$\frac{\mathcal{E}}{l} = \frac{1}{\pi^2}\sum_{n=1,2,3}^{\infty} \frac{\sin^2\frac{n\pi}{3}\left[\left(\frac{F_1}{P_E}\right)^2 - 2(-1)^n\frac{F_1F_2}{P_E^2} + \left(\frac{F_2}{P_E}\right)^2\right]}{n^2\left[\frac{P}{P_E} - n^2\right]^2}$$

$$\frac{\mathcal{E}}{l} = C\left(\frac{F_1}{P_E}\right)^2 + D\left(\frac{F_1F_2}{P_E^2}\right) + C\left(\frac{F_2}{F_E}\right)^2$$

where

$$C = \frac{1}{\pi^2}\sum_{n=1,2,3}^{\infty} \frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]^2} \; ; \quad D = -\frac{2}{\pi^2}\sum_{n=1,2,3}^{\infty} (-1)^n \frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]^2}$$

In case of symmetrical solution, $n = 2m = 0$,

$$\therefore \quad \frac{\delta}{l} = H\left(\frac{F}{P_E}\right) \quad \text{where} \quad H = \frac{4}{\pi^2}\sum_{n=1,3,5}^{\infty} \frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

$$\left(\frac{a_n}{l}\right)_{n=2m+1} = \frac{4}{\pi^2} \frac{\sin\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]}\left(\frac{F_1}{P_E}\right); \quad \left(\frac{a_n}{l}\right)_{n=2m} = 0$$

$$\frac{\varepsilon}{l} = \frac{4}{\pi^2}\left(\frac{F}{P_E}\right)^2 \sum_{n=1,3,5}^{\infty} \frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]^2}$$

$$\left(\frac{F}{P_E}\right) \Big/ \left(\frac{\delta}{l}\right) = \frac{1}{\frac{4}{\pi^2}\sum_{n=1,3,5}^{\infty} \frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]}}$$

$$A = \frac{2}{\pi^2}\sum_{n=1,2,3}^{\infty} \frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]} = \frac{3}{2\pi^2}\left\{\sum_{n=1,2,3}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E} - n^2\right]} - \sum_{m=1,2,3}^{\infty} \frac{1}{9m^2\left[\frac{P}{P_E} - 9m^2\right]}\right\}$$

$$= \frac{3}{2\pi^2}\left\{\sum_{n=1,2,3}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E} - n^2\right]} - \frac{1}{81}\sum_{m=1,2,3}^{\infty} \frac{1}{m^2\left[\frac{P}{9P_E} - m^2\right]}\right\}$$

$$A = \frac{3}{2\pi^2}\left\{\frac{1}{\left(\frac{P}{P_E}\right)}\left[\frac{\pi^2}{6} + \sum_{n=1}^{3} \frac{1}{\left(\frac{P}{P_E}\right) - n^2} - \frac{1}{2\sqrt{\frac{P}{P_E}}}\left(\psi\left(4 + \sqrt{\frac{P}{P_E}}\right) - \psi\left(4 - \sqrt{\frac{P}{P_E}}\right)\right)\right]\right.$$
$$\left. - \frac{1}{9\left(\frac{P}{P_E}\right)}\left[\frac{\pi^2}{6} - \frac{1}{\frac{2}{3}\sqrt{\frac{P}{P_E}}}\left(\psi\left(1 + \frac{1}{3}\sqrt{\frac{P}{P_E}}\right) - \psi\left(1 - \frac{1}{3}\sqrt{\frac{P}{P_E}}\right)\right)\right]\right\}$$

$$H = \frac{4}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{\sin^2 \frac{n\pi}{3}}{n^2\left[\frac{P}{P_E} - n^2\right]} = \frac{3}{\pi^2} \left\{ \sum_{n=1,3,5}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E} - n^2\right]} - \frac{1}{81} \sum_{m=1,3,5}^{\infty} \frac{1}{m^2\left[\frac{P}{9P_E} - m^2\right]} \right\}$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E} - n^2\right]} = \frac{1}{\frac{P}{P_E}} \sum_{n=1,3,5}^{\infty} \left\{ \frac{1}{n^2} + \frac{1}{\frac{P}{P_E} - n^2} \right\}$$

$$= \frac{1}{\frac{P}{P_E}} \left\{ \frac{\pi^2}{8} + \sum_{n=1,3}^{3} \frac{1}{\frac{P}{P_E} - n^2} - \frac{1}{2\sqrt{\frac{P}{P_E}}} \sum_{n=5,7}^{\infty} \left[\frac{1}{n - \sqrt{\frac{P}{P_E}}} - \frac{1}{n + \sqrt{\frac{P}{P_E}}} \right] \right\}$$

$$\sum_{n=5,7}^{\infty} \frac{1}{n - \sqrt{\frac{P}{P_E}}} = \sum_{n=5,7}^{\infty} \int_0^{\infty} e^{-x\left(n - \sqrt{\frac{P}{P_E}}\right)} dx$$

$$= \int_0^{\infty} e^{-x\left(5 - \sqrt{\frac{P}{P_E}}\right)} \sum_{n=0}^{\infty} e^{-2xn} \, dx$$

$$= \int_0^{\infty} e^{-x\left(5 - \sqrt{\frac{P}{P_E}}\right)} \frac{1}{1 - e^{-2x}} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-\xi\left(\frac{5}{2} - \frac{1}{2}\sqrt{\frac{P}{P_E}}\right)} \frac{d\xi}{1 - e^{-\xi}}$$

$$\therefore \sum_{n=1,3,5}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]} = \frac{1}{\frac{P}{P_E}}\left\{\frac{\pi^2}{8} + \sum_{n=1,3}^{3} \frac{1}{\frac{P}{P_E}-n^2} - \frac{1}{4\sqrt{\frac{P}{P_E}}}\left[\psi\left(\frac{5}{2}+\frac{1}{2}\sqrt{\frac{P}{P_E}}\right) - \psi\left(\frac{5}{2}-\frac{1}{2}\sqrt{\frac{P}{P_E}}\right)\right]\right\}$$

$$H = \frac{3}{\pi^2}\left[\frac{1}{\frac{P}{P_E}}\left\{\frac{\pi^2}{8} + \sum_{n=1,3}^{3}\frac{1}{\frac{P}{P_E}-n^2} - \frac{1}{4\sqrt{\frac{P}{P_E}}}\left[\psi\left(\frac{5+\sqrt{\frac{P}{P_E}}}{2}\right) - \psi\left(\frac{5-\sqrt{\frac{P}{P_E}}}{2}\right)\right]\right\} - \frac{1}{9\frac{P}{P_E}}\left\{\frac{\pi^2}{8} - \frac{1}{\frac{4}{3}\sqrt{\frac{P}{P_E}}}\left[\psi\left(\frac{1+\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right) - \psi\left(\frac{1-\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right)\right]\right\}\right]$$

By analytical continuation

$$A = \frac{1}{\frac{P}{P_E}}\left[\frac{1}{4} - \frac{3}{4\pi^2\sqrt{\frac{P}{P_E}}}\left\{\psi\left(1+\sqrt{\frac{P}{P_E}}\right) - \psi\left(1-\sqrt{\frac{P}{P_E}}\right)\right\} - \frac{1}{36} + \frac{1}{4\pi^2\sqrt{\frac{P}{P_E}}}\left\{\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) - \psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right)\right\}\right]$$

$$A = \frac{1}{\frac{P}{P_E}}\left[\frac{2}{9} + \frac{1}{4\pi^2\sqrt{\frac{P}{P_E}}}\left\{3\psi\left(1-\sqrt{\frac{P}{P_E}}\right) + \psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) - 3\psi\left(1+\sqrt{\frac{P}{P_E}}\right) - \psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right)\right\}\right]$$

$$H = \frac{1}{\frac{P}{P_E}}\left[\frac{1}{3} + \frac{1}{4\pi^2\sqrt{\frac{P}{P_E}}}\left\{3\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{2}\right) + \psi\left(\frac{1+\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right) - 3\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{2}\right) - \psi\left(\frac{1-\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right)\right\}\right] \tag{9}$$

$\frac{P}{P_E} = 8.41$; $\sqrt{\frac{P}{P_E}} = 2.9$ $\qquad$ $4\pi^2 = 39.47840$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-1.9) = -8.78633$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.96667) = 0.40106$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(3.9) = 1.22733$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.033333)$

$= \psi(1.03333) - \frac{1}{0.033333}$

$= -0.52368 - 30 = -30.52368$

$$\underline{A(8.41)} = \frac{1}{8.41}\left[0.222222 + \frac{1}{4\pi^2 \times 2.9} \times 0.88376\right] = \frac{0.23057}{8.41} = \underline{+0.027416}$$

$\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(-0.95) = -19.44521$; $\psi\left(\frac{1+\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(0.983333)$

$= -0.60500$

$\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(1.95) = 0.39002$; $\psi\left(\frac{1-\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right) = \psi\left(\frac{0.033333}{2}\right)$

$= \psi(1.016667) - 60 = -60.55012$

$$\underline{H(8.41)} = \frac{1}{8.41}\left[0.333333 + \frac{1}{4\pi^2 \times 2.9} \times 0.43943\right] = \underline{0.040092}$$

$\underline{B = H - A = 0.012676}$

$\frac{P}{P_E} = \underline{7.84}$; $\sqrt{\frac{P}{P_E}} = \underline{2.8}$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-1.8) = -3.48348$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.93333) = +0.37886$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(3.8) = 1.19769$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.066667)$

$= \psi(1.066667) - 15 = -15.47259$

$\underline{A(7.84) = 0.030431}$;

$\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(-0.9) = -9.31264$; $\psi\left(\frac{1+\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(0.966667)$

$= -0.63345$

$\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(1.9) = 0.35618$; $\psi\left(\frac{1-\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(0.033333)$

$= -30.52368$

$\underline{H(7.84) = 0.043537}$

$\underline{B(7.84) = 0.013106}$

$\sqrt{\frac{P}{P_E}} = \underline{2.7}$; $\frac{P}{P_E} = \underline{7.29}$;

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-1.7) = -1.48572$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.9) = 0.35618$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(3.7) = 1.16715$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.1) = -10.42375$

$\underline{A = 0.034114}$

$\psi(-0.85) = -5.84452$; $\psi(0.95) = -0.66261$

$\psi(1.85) = 0.32120$; $\psi(0.05) = -20.49784$

$\underline{H = 0.047446}$; $\underline{B = 0.013332}$

$\sqrt{\frac{P}{P_E}} = \underline{2.6}, \quad \frac{P}{P_E} = \underline{6.76}$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-1.6) = -0.26972 \; ; \; \psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.86667) = 0.33299$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(3.6) = 1.13566 \; ; \; \psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.13333) = -7.87698$

$\underline{A = 0.038629}$

$\psi(-0.8) = -4.03904 \; ; \; \psi(0.93333) = -0.69259$

$\psi(1.8) = 0.28499 \; ; \; \psi(0.066667) = -15.47259$

$H = \underline{0.051914} \; ; \quad B = \underline{0.013285}$

$\sqrt{\frac{P}{P_E}} = \underline{2.5} \; ; \quad \frac{P}{P_E} = \underline{6.25}$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-1.5) = 0.70316 \; ; \; \psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.83333) = 0.30927$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(3.5) = 1.10316 \; ; \; \psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.166667) = -6.33212$

$A = \underline{0.044376}$

$\psi(-0.75) = -2.89412 \; ; \; \psi(0.916667) = -0.72333$

$\psi(1.75) = 0.24747 \; ; \; \psi(0.083333) = -12.44790.$

$H = \underline{0.057061} \qquad B = \underline{0.012685}$

$\sqrt{\frac{P}{P_E}} = \underline{2.4}$; $\frac{P}{P_E} = \underline{5.76}$

$\psi(1-\sqrt{\frac{P}{P_E}}) = \psi(-1.4) = 1.67367$; $\psi(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}) = \psi(1.800) = 0.28499$

$\psi(1+\sqrt{\frac{P}{P_E}}) = \psi(+3.4) = 1.06957$; $\psi(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}) = \psi(0.20) = -5.28904$

$A = \underline{0.052115}$

$\psi(-0.7) = -2.07395$; $\psi(0.90) = -0.75493$

$\psi(1.7) = 0.20855$; $\psi(0.10) = -10.42375$

$H = \underline{0.063040}$, $B = \underline{0.010925}$

$\sqrt{\frac{P}{P_E}} = \underline{2.3}$; $\frac{P}{P_E} = \underline{5.29}$

$\psi(1-\sqrt{\frac{P}{P_E}}) = \psi(-1.3) = 2.88254$; $\psi(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}) = \psi(1.766667) = 0.26013$

$\psi(1+\sqrt{\frac{P}{P_E}}) = \psi(3.3) = 1.03482$; $\psi(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}) = \psi(0.23333) = -4.53337$

$A = \underline{0.063527}$

$\psi(-0.65) = -1.43261$; $\psi(0.883333) = -0.78746$

$\psi(1.65) = 0.16811$; $\psi(0.1166667) = -8.97152$

$\underline{H = 0.070053}$ $B = \underline{0.006526}$

$\sqrt{\frac{P}{P_E}} = 2.2$; ~~$\frac{P}{P_E} = 4.84$~~

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-1.2) = 4.86832$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.73333) = 0.23466$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(3.2) = 0.99884$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.266667) = -3.95768$

$A = \underline{0.083502}$

$\psi(-0.6) = -0.89472$; $\psi(0.866667) = -0.82086$

$\psi(1.6) = 0.12605$; $\psi(0.13333) = -7.87698$

$H = \underline{0.078370}$ $B = \underline{-0.005132}$

$\sqrt{\frac{P}{P_E}} = 2.1$, $\frac{P}{P_E} = 4.41$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-1.1) = 10.15416$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.7) = 0.20855$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(3.1) = 0.96153$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.3) = -3.50252$

$A = \underline{0.135971}$

$\psi(-0.55) = -0.41536$; $\psi(0.85) = -0.85527$

$\psi(1.55) = 0.08222$; $\psi(0.15) = -7.02099$

$H = \underline{0.088317}$ $B = -\underline{0.047604}$

$\sqrt{\frac{P}{P_E}} = \underline{2.0}$ $\frac{P}{P_E} = \underline{4.00}$, $\underline{A - \infty}$

$\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(-0.5) = 0.03649$; $\psi\left(\frac{1+\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(0.83333) = -0.89073$

$\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(1.5) = 0.03649$; $\psi\left(\frac{1-\frac{1}{3}\sqrt{\frac{P}{P_E}}}{2}\right) = \psi(0.166667) = -6.33212$

$H = \underline{0.100563}$

$\sqrt{\frac{P}{P_E}} = \underline{1.9}$; $\frac{P}{P_E} = \underline{3.61}$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-0.9) = -9.31264$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.63333) = 0.15427$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(2.9) = 0.88250$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.36667) = -2.82339$

$\underline{A = -0.040398}$

$\psi(-0.45) = 0.48626$; $\psi(0.8166667) = -0.92229$

$\psi(1.45) = -0.01132$; $\psi(0.183333) = -5.76492$

$H = \underline{0.115712}$ $B = \underline{0.156110}$

$\sqrt{\frac{P}{P_E}} = \underline{1.8}$; $\frac{P}{P_E} = \underline{3.24}$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-0.8) = -4.03904$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.6) = 0.12605$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(2.8) = 0.84055$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.4) = -2.56138$

$A = +0.\underline{016678}$

$\psi(-0.4) = +0.95938$; $\psi(0.8) = -0.96500$

$\psi(1.4) = -0.06138$; $\psi(0.2) = -5.28904$

$H = \underline{0.134962}$ $B = \underline{0.118284}$

$\sqrt{\frac{P}{P_E}} = \underline{1.7}\ ; \quad \frac{P}{P_E} = \underline{2.89}$

$\psi(1-\sqrt{\frac{P}{P_E}}) = \psi(-0.7) = -2.07395\ ; \quad \psi(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}) = \psi(1.56667) = 0.09703$

$\psi(1+\sqrt{\frac{P}{P_E}}) = \psi(2.7) = 0.79678\ ; \quad \psi(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}) = \psi(0.43333) = -2.33543$

$A = \underline{0.045032}$

$\psi(-0.35) = 1.48679\ ; \quad \psi(0.783333) = -1.00397$

$\psi(1.35) = -0.11393\ ; \quad \psi(0.216667) = -4.88349$

$\underline{H = 0.160101} \qquad \underline{B = 0.115069}$

$\sqrt{\frac{P}{P_E}} = \underline{1.6}\ ; \quad \frac{P}{P_E} = \underline{2.56}$

$\psi(1-\sqrt{\frac{P}{P_E}}) = \psi(-0.6) = -0.89472\ ; \quad \psi(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}) = \psi(1.53333) = 0.06720$

$\psi(1+\sqrt{\frac{P}{P_E}}) = \psi(2.6) = 0.25105\ ; \quad \psi(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}) = \psi(0.466667) = -2.13800$

$A = \underline{0.069910}$

$\psi(-0.3) = +2.11331\ ; \quad \psi(0.766667) = -1.04422$

$\psi(1.3) = -0.16919\ ; \quad \psi(0.233333) = -4.53330$

$H = \underline{0.194131} \qquad B = \underline{0.124221}$

$\sqrt{\frac{P}{P_E}} = \underline{1.5}$; $\frac{P}{P_E} = \underline{2.25}$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-0.5) = +0.03649$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.5) = +0.03649$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(2.5) = +0.70316$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.5) = -1.96351$

$A = \underline{0.098765}$

$\psi(-0.25) = +2.91414$; $\psi(0.75) = -1.08586$

$\psi(1.25) = -0.22745$; $\psi(0.25) = -4.22745$

$H = \underline{0.242462}$ $\quad B = \underline{0.143197}$

$\sqrt{\frac{P}{P_E}} = \underline{1.4}$; $\frac{P}{P_E} = \underline{1.96}$

$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi(-0.4) = 0.95758$; $\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(1.466667) = +0.00485$

$\psi\left(1+\sqrt{\frac{P}{P_E}}\right) = \psi(2.4) = +0.65290$; $\psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi(0.53333) = -1.80740$

$A = \underline{0.138599}$

$\frac{\pi}{2}\sqrt{\frac{P}{P_E}} = \frac{\pi}{2}(1.4) = \frac{\pi}{2} + \frac{\pi}{2}0.4$; $\quad \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} = -\cot\frac{\pi}{2}\times 0.4$

$= -\cot 0.62832 = -\frac{0.80902}{0.58779}$

$= -1.37638$

$\frac{\pi}{6}\sqrt{\frac{P}{P_E}} = 0.73304$; $\quad \tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}} = 1/\frac{0.74314}{0.66913} =$

$\underline{H = 0.315928}$ $\quad B = \underline{0.177329}$

$\pi = 3.141593$

$\frac{\pi}{3} = 1.047198$

$\sqrt{\frac{P}{P_E}} = \underline{1.3}$; $\frac{P}{P_E} = \underline{1.69}$

$\cot \pi \sqrt{\frac{P}{P_E}} = \cot \pi\, 0.3 = \cot 0.942478 = \frac{0.587785}{0.809017} = 0.726542$

$\cot \frac{\pi}{3} \sqrt{\frac{P}{P_E}} = \cot 1.361357 = \frac{0.207911}{0.978148} = 0.212556$

$\underline{A = 0.202741}$

$\tan \frac{\pi}{2} \sqrt{\frac{P}{P_E}} = -\cot 0.471238 = -\frac{0.891007}{0.453990} = -1.962614$

$\tan \frac{\pi}{6} \sqrt{\frac{P}{P_E}} = \tan 0.680679 = \frac{0.629321}{0.777145} = 0.809786$

$H = \underline{0.439633}$; $B = \underline{0.337092}$

$\sqrt{\frac{P}{P_E}} = \underline{1.2}$; $\frac{P}{P_E} = \underline{1.44}$

$\cot \pi \sqrt{\frac{P}{P_E}} = \cot 0.628318 = 1.37638$

$\cot \frac{\pi}{3} \sqrt{\frac{P}{P_E}} = \cot 1.256638 = \frac{0.309016}{0.951057} = 0.324918$

$\underline{A = 0.329512}$

$\tan \frac{\pi}{2} \sqrt{\frac{P}{P_E}} = -\cot 0.314159 = \frac{-0.951057}{0.309017} = -3.077685$

$\tan \frac{\pi}{6} \sqrt{\frac{P}{P_E}} = \tan 0.628318 = 0.726546$

$H = \underline{0.690139}$ $B = \underline{0.360627}$

When $\sqrt{\frac{P}{P_E}} > 1$

$$\psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \psi\left[-\left(\sqrt{\frac{P}{P_E}}-1\right)\right] = \psi\left(\sqrt{\frac{P}{P_E}}\right) + \pi \cot \pi\left(\sqrt{\frac{P}{P_E}}-1\right)$$

$$= \psi\left(\sqrt{\frac{P}{P_E}}\right) + \pi \cot \pi\sqrt{\frac{P}{P_E}} = \psi\left(1+\sqrt{\frac{P}{P_E}}\right) - \frac{1}{\sqrt{\frac{P}{P_E}}} + \pi \cot \pi\sqrt{\frac{P}{P_E}}$$

$$\therefore \quad \psi\left(1+\sqrt{\frac{P}{P_E}}\right) - \psi\left(1-\sqrt{\frac{P}{P_E}}\right) = \frac{1}{\sqrt{\frac{P}{P_E}}} - \pi \cot \pi\sqrt{\frac{P}{P_E}}$$

$$\psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi\left(\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) + \frac{3}{\sqrt{\frac{P}{P_E}}}$$

$$\therefore \quad \psi\left(1+\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) - \psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) = \psi\left(\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) - \psi\left(1-\frac{1}{3}\sqrt{\frac{P}{P_E}}\right) + \frac{3}{\sqrt{\frac{P}{P_E}}}$$

$$= \frac{3}{\sqrt{\frac{P}{P_E}}} - \pi \cot \pi\frac{\sqrt{\frac{P}{P_E}}}{3}$$

$$\therefore \quad A = \frac{1}{\frac{P}{P_E}}\left[\frac{2}{9} + \frac{1}{4\pi^2\sqrt{\frac{P}{P_E}}}\left\{3\pi \cot \pi\sqrt{\frac{P}{P_E}} - \frac{3}{\sqrt{\frac{P}{P_E}}} + \frac{3}{\sqrt{\frac{P}{P_E}}} - \pi \cot \pi\frac{\sqrt{\frac{P}{P_E}}}{3}\right\}\right]$$

$$\boxed{A = \frac{1}{\frac{P}{P_E}}\left[\frac{2}{9} + \frac{1}{4\pi\sqrt{\frac{P}{P_E}}}\left\{3 \cot \pi\sqrt{\frac{P}{P_E}} - \cot \frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right\}\right]}$$

$$\psi\left(\frac{1-\sqrt{\frac{p}{p_E}}}{2}\right)=\psi\left[-\left(\frac{\sqrt{\frac{p}{p_E}}-1}{2}\right)\right]=\psi\left(\frac{\sqrt{\frac{p}{p_E}}+1}{2}\right)+\pi\cot\pi\left(\frac{\sqrt{\frac{p}{p_E}}-1}{2}\right) \quad 19$$

$$=\psi\left(\frac{1+\sqrt{\frac{p}{p_E}}}{2}\right)-\pi\tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}$$

$$\therefore\quad \psi\left(\frac{1-\sqrt{\frac{p}{p_E}}}{2}\right)-\psi\left(\frac{1+\sqrt{\frac{p}{p_E}}}{2}\right)=-\pi\tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}$$

$$\psi\left(\frac{1-\frac{1}{3}\sqrt{\frac{p}{p_E}}}{2}\right)=\psi\left[1-\frac{1+\frac{1}{3}\sqrt{\frac{p}{p_E}}}{2}\right]$$

$$\therefore\quad \psi\left(\frac{1-\frac{1}{3}\sqrt{\frac{p}{p_E}}}{2}\right)-\psi\left(\frac{1+\frac{1}{3}\sqrt{\frac{p}{p_E}}}{2}\right)=\pi\cot\pi\left(\frac{1+\frac{1}{3}\sqrt{\frac{p}{p_E}}}{2}\right)$$

$$=-\pi\tan\frac{\pi}{6}\sqrt{\frac{p}{p_E}}$$

$$\boxed{H=\frac{1}{\frac{p}{p_E}}\left[\frac{1}{3}-\frac{1}{4\pi\sqrt{\frac{p}{p_E}}}\left\{3\tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}-\tan\frac{\pi}{6}\sqrt{\frac{p}{p_E}}\right\}\right]}$$

$$\boxed{B=\frac{1}{\frac{p}{p_E}}\left[\frac{1}{9}-\frac{1}{4\pi\sqrt{\frac{p}{p_E}}}\left\{3\left(\tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}+\cot\pi\sqrt{\frac{p}{p_E}}\right)-\left(\tan\frac{\pi}{6}\sqrt{\frac{p}{p_E}}+\cot\frac{\pi}{3}\sqrt{\frac{p}{p_E}}\right)\right\}\right]}$$

$\sqrt{\frac{P}{P_E}} = \underline{1.1}$; $\frac{P}{P_E} = \underline{1.21}$;

$\cot \pi \sqrt{\frac{P}{P_E}} = \cot 0.314159 = 3.077685$

$\cot \frac{\pi}{3} \sqrt{\frac{P}{P_E}} = \cot 1.151918 = \frac{0.406736}{0.913545} = 0.445228$

$A = \underline{0.709059}$

$\tan \frac{\pi}{2} \sqrt{\frac{P}{P_E}} = -\cot 0.157079\dot{6} = -\frac{0.987688}{0.156435} = -6.313728$

$\tan \frac{\pi}{6} \sqrt{\frac{P}{P_E}} = \tan 0.575959 = \frac{0.544640}{0.838671} = 0.649408$

$H = \underline{1.446759}$ $\quad B = \underline{0.737700}$

$\sqrt{\frac{P}{P_E}} = \underline{1.0}$; $\frac{P}{P_E} = \underline{1.00}$

$A = +\infty, -\infty$; $\quad H = \infty$; $\quad B = \infty, = A$!!!

Let us investigate the case $\sqrt{\frac{P}{P_E}} = \underline{3 - \varepsilon}$; $\quad \underline{\varepsilon \ll 1}$

Then

$$3 \cot \pi \sqrt{\frac{P}{P_E}} - \cot \frac{\pi}{3} \sqrt{\frac{P}{P_E}} = 3 \cot \pi (3 - \varepsilon) - \cot \pi \left(1 - \frac{\varepsilon}{3}\right)$$

$$= -3 \cot \pi\varepsilon + \cot \frac{\pi\varepsilon}{3} = -3 \frac{\cos \pi\varepsilon}{\sin \pi\varepsilon} + \frac{\cos \frac{\pi\varepsilon}{3}}{\sin \frac{\pi\varepsilon}{3}}$$

$$= -3 \frac{1 - \frac{1}{2!}(\pi\varepsilon)^2 + \cdots}{\pi\varepsilon\left[1 - \frac{1}{3!}(\pi\varepsilon)^2 + \cdots\right]} + 3 \frac{1 - \frac{1}{2!}\left(\frac{\pi\varepsilon}{3}\right)^2 + \cdots}{\pi\varepsilon\left[1 - \frac{1}{3!}\left(\frac{\pi\varepsilon}{3}\right)^2 + \cdots\right]}$$

$$= \frac{3}{\pi\varepsilon}\left[-\left(1 - \frac{1}{2!}(\pi\varepsilon)^2 + \cdots\right)\left(1 + \frac{1}{3!}(\pi\varepsilon)^2 + \cdots\right) + \left(1 - \frac{1}{2!}\left(\frac{\pi\varepsilon}{3}\right)^2 \cdots\right)\left(1 + \frac{1}{3!}\left(\frac{\pi\varepsilon}{3}\right)^2 \cdots\right)\right]$$

$$= O(\pi\varepsilon) \; -> 0.$$

$\sqrt{\frac{P}{P_E}} = 0.9$; $\frac{P}{P_E} = 0.81$

$\cot \pi\sqrt{\frac{P}{P_E}} = -\tan 1.256637 = -\frac{0.951056}{0.309019} = -3.077662$

$\cot \frac{\pi}{3}\sqrt{\frac{P}{P_E}} = \cot 0.942478 = \frac{0.587785}{0.809017} = 0.726542$

$A = -0.812832$

$\tan \frac{\pi}{2}\sqrt{\frac{P}{P_E}} = 6.313728$

$\tan \frac{\pi}{6}\sqrt{\frac{P}{P_E}} = \tan 0.471239 = 0.509525$

$H = -1.600473$ $B = -0.787641$

$\sqrt{\frac{P}{P_E}} = 0.8$; $\frac{P}{P_E} = 0.64$

$\cot \pi\sqrt{\frac{P}{P_E}} = -\tan 0.942478 = -1.376383$

$\cot \frac{\pi}{3}\sqrt{\frac{P}{P_E}} = \cot 0.837758 = \frac{0.669131}{0.743145} = 0.900404$

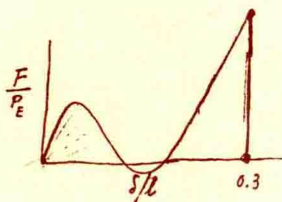
$A = -0.434495$

$\tan \frac{\pi}{2}\sqrt{\frac{P}{P_E}} = \tan 1.256637 = 3.077662$

$\tan \frac{\pi}{6}\sqrt{\frac{P}{P_E}} = \tan 0.418879 = \frac{0.406737}{0.913546} = 0.445228$

$H = -0.845002$ $B = -0.410507$

$\sqrt{\frac{P}{P_E}}$	$\frac{P}{P_E}$	A	B	H	1/H
3.0	9.0	0.024691	0.012346	0.037037	27.000
2.9	8.41	0.027416	0.012676	0.040092	24.943
2.8	7.84	0.030431	0.013106	0.043537	22.969
2.7	7.29	0.034114	0.013332	0.047446	21.077
2.6	6.76	0.038629	0.013285	0.051914	19.263
2.5	6.25	0.044376	0.012685	0.057061	17.525
2.4	5.76	0.052115	0.010925	0.063040	15.863
2.3	5.29	0.063527	0.006526	0.070053	14.275
2.2	4.84	0.083502	−0.005132	0.078370	12.760
2.1	4.41	0.135971	−0.047604	0.088367	11.316
2.0	4.00	∞ $(-\infty)$	$-\infty$ $(+\infty)$	0.100563	9.9440
1.9	3.61	−0.040398	0.156110	0.115712	8.6421
1.8	3.24	+0.016678	0.118284	0.134962	7.4095
1.7	2.89	+0.045032	0.115069	0.160101	6.2461
1.6	2.56	+0.069910	0.124221	0.194131	5.1512
1.5	2.25	+0.098765	0.143697	0.242462	4.1244
1.4	1.96	+0.138599	0.177329	0.315928	3.1653
1.3	1.69	+0.202741	0.237092	0.439833	2.2736
1.2	1.44	+0.329512	0.360627	0.690139	1.44898
1.1	1.21	+0.709059	0.737700	1.446759	0.69120
1.0	1.00	$+\infty$ $(-\infty)$	$+\infty$ $(-\infty)$	∞	0
0.9	0.81	−0.812832	−0.787641	−1.600473	−0.62482
0.8	0.64	−0.434495	−0.410507	−0.845002	−1.18343
0.7	0.49				
0.6	0.36				



$$\frac{F}{P_E} \Big/ \delta/l = a - bx + cx^2 = f$$

$$a = 27.000$$

$$f = 27.000 - bx + cx^2$$

$$\frac{\partial f}{\partial x} = -b + 2cx = 0 \quad ; \quad x = \frac{b}{2c} = 0.1$$

$$\therefore \quad b = 0.2c \qquad c = 5b$$

$$-1.2 = 27.000 - 0.1b + 0.01c = 27.000 - 0.1b + 0.05b$$

$$= 27.000 - 0.050b$$

$$b = \frac{27.00 + 1.2}{0.05} = \frac{28.2}{0.05} = 564$$

$$\boxed{\frac{\frac{F}{P_E}}{\delta/l} = 27.000 - 564.000\left(\frac{\delta}{l}\right) + 2820.00\left(\frac{\delta}{l}\right)^2 = \frac{1}{H}}$$

$$\frac{P}{P_E} = 9.00 \qquad \xi = \frac{\delta}{l} = 0 \; ; \; \text{or} \qquad \xi = \frac{564}{2820} = 0.20000$$

$$\frac{P}{P_E} = 8.41$$

$$2820\,\xi^2 - 564\,\xi + 2.057 = 0$$

$$\xi^2 - 0.20000\,\xi + 0.0007294 = 0$$

$$\xi = 0.1 \pm \sqrt{0.01 - 0.0007294} = 0.1 \pm 0.096284 = \begin{matrix} 0.196284 \\ 0.003716 \end{matrix}$$

①	②	③	④	⑤	⑥	⑦	⑧
P/P_E	$27-\frac{1}{H}$	②/2820	0.01 − ③	$\sqrt{④}$	ξ_1	ξ_2	
7.84	4.031	0.0014294	0.0085706	0.092578	0.192578	0.007422	
7.29	5.923	0.0021003	0.0078997	0.088880	0.188880	0.011120	
6.76	7.737	0.0027436	0.0072564	0.085184	0.185184	0.014816	
6.25	9.475	0.0033599	0.0066401	0.081487	0.181487	0.018513	
5.76	11.137	0.0039493	0.0060507	0.077786	0.177786	0.022214	
5.29	12.725		0.0054876	0.074079	0.174079	0.025921	
4.84	14.240		0.0049504	0.070360	0.170360	0.029640	
4.41	15.684		0.0044383	0.066621	0.166621	0.033379	
4.00	17.056		0.0039518	0.062863	0.162863	0.037137	
3.61	18.358		0.0034901	0.059077	0.159077	0.040923	
3.24	19.590		0.0030532	0.055256	0.155256	0.044744	
2.89	20.954		0.0025695	0.050690	0.150690	0.049310	
2.56	21.849		0.0022521	0.047456	0.147456	0.052544	
2.25	22.876		0.0018879	0.043450	0.143450	0.056550	
1.96	23.835		0.0015479	0.039343	0.139343	0.060657	
1.69	24.726		0.0012319	0.035099	0.135099	0.064901	
1.44	25.551		0.0009394	0.030650	0.130650	0.069350	
1.21	26.309		0.0006706	0.025895	0.125895	0.074105	
1.00	27.000		0.0004255	0.020628	0.120628	0.079372	
0.81	27.625		0.0002039	0.014279	0.114279	0.085721	
0.64	28.183		0.0000060	0.002448	0.102448	0.097552	
0.49							

$$\sum_{n=1,3,5}^{\infty} \frac{\sin^2 \frac{n\pi}{3}}{n^2\left[\frac{P}{P_E}-n^2\right]^2} = \frac{3}{4}\left[\sum_{n=1,3,5}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]^2} - \frac{1}{729}\sum_{m=1,3,5}^{\infty} \frac{1}{m^2\left[\frac{P}{9P_E}-m^2\right]^2}\right]$$

But $$\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]^2} = \frac{A}{n^2} + \frac{B}{\left[\frac{P}{P_E}-n^2\right]^2} + \frac{C}{\left[\frac{P}{P_E}-n^2\right]}$$

$$= \frac{A\left[\frac{P}{P_E}-n^2\right]^2 + Bn^2 + Cn^2\left[\frac{P}{P_E}-n^2\right]}{n^2\left[\frac{P}{P_E}-n^2\right]^2}$$

$$= \frac{A\left(\frac{P}{P_E}\right)^2 + n^2\left[-2\left(\frac{P}{P_E}\right)A + B + C\left(\frac{P}{P_E}\right)\right] + n^4\left[A-C\right]}{n^2\left[\frac{P}{P_E}-n^2\right]^2}$$

$$A = C; \qquad B = \frac{P}{P_E}A, \qquad A = \frac{1}{\left(\frac{P}{P_E}\right)^2}$$

$$\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]^2} = \frac{1}{\left(\frac{P}{P_E}\right)^2}\frac{1}{n^2} + \frac{1}{\left(\frac{P}{P_E}\right)}\frac{1}{\left[\frac{P}{P_E}-n^2\right]^2} + \frac{1}{\left(\frac{P}{P_E}\right)^2}\frac{1}{\left[\frac{P}{P_E}-n^2\right]}$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left(\frac{1}{n^2} + \frac{1}{\frac{P}{P_E}-n^2}\right) + \frac{1}{\left(\frac{P}{P_E}\right)}\frac{1}{\left[\frac{P}{P_E}-n^2\right]^2}$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left[\frac{1}{n^2} + \frac{1}{\frac{P}{P_E}-n^2}\right] - \frac{1}{\left(\frac{P}{P_E}\right)}\frac{\partial}{\partial\left(\frac{P}{P_E}\right)}\left\{\frac{1}{\left[\frac{P}{P_E}-n^2\right]}\right\}$$

$$\therefore \sum_{n=1,3,5}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]^2} = \frac{1}{\left(\frac{P}{P_E}\right)^2}\left[\frac{\pi^2}{8} - \frac{1}{4\sqrt{\frac{P}{P_E}}}\left\{\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{2}\right) - \psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{2}\right)\right\}\right]$$

$$+ \frac{1}{\left(\frac{P}{P_E}\right)}\frac{\partial}{\partial\left(\frac{P}{P_E}\right)}\left[\frac{1}{4\sqrt{\frac{P}{P_E}}}\left\{\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{2}\right) - \psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{2}\right)\right\}\right]$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left[\frac{\pi^2}{8} - \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right] + \frac{1}{\frac{P}{P_E}}\frac{\partial}{\partial\left(\frac{P}{P_E}\right)}\left[\frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left[\frac{\pi^2}{8} - \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{\pi}{8}\sqrt{\frac{P}{P_E}}\cdot\frac{\partial}{\partial\sqrt{\frac{P}{P_E}}}\left\{\frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\sqrt{\frac{P}{P_E}}}\right\}\right]$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left[\frac{\pi^2}{8} - \frac{3\pi}{8\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{\pi^2}{16}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left[\frac{\pi^2}{8}\left(1+\frac{1}{2}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right) - \frac{3\pi}{8\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$\frac{\varepsilon}{l} = 3\left(\frac{F}{P_E}\right)^2\frac{1}{\left(\frac{P}{P_E}\right)^2}\left[\frac{1}{8}\left(\frac{3}{2}+\frac{1}{2}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right) - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right.$$

$$\left. - \frac{1}{9}\left\{\frac{1}{8}\left(\frac{3}{2}+\frac{1}{2}\tan^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right) - \frac{9}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right\}\right]$$

$$\boxed{\frac{\varepsilon}{l} = \left(\frac{\frac{F}{P_E}}{\frac{P}{P_E}}\right)^2\left[\frac{1}{2} + \frac{3}{16}\left(\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\tan^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right) - \frac{9}{8\pi\sqrt{\frac{P}{P_E}}}\left(\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)\right]}$$

28.

$$\frac{F}{P_E} = \frac{\delta}{\ell}\Big/ H = \left(\frac{\delta}{\ell}\right)\cdot\left(\frac{P}{P_E}\right)\frac{1}{\frac{1}{3} - \frac{1}{4\pi\sqrt{\frac{P}{P_E}}}\left\{3\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right\}}$$

$$\boxed{\frac{\varepsilon}{\ell} = \left(\frac{\delta}{\ell}\right)^2\cdot\frac{\frac{1}{2} + \frac{3}{16}\left(\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\tan^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right) - \frac{9}{8\pi\sqrt{\frac{P}{P_E}}}\left(\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)}{\left[\frac{1}{3} - \frac{3}{4\pi\sqrt{\frac{P}{P_E}}}\left(\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)\right]^2}}$$

$$\alpha = \frac{1}{2} + \frac{3}{16}\left(\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\tan^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right) - \frac{9}{8\pi\sqrt{\frac{P}{P_E}}}\left(\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)$$

$$\beta = \left[\frac{1}{3} - \frac{3}{4\pi\sqrt{\frac{P}{P_E}}}\left(\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)\right]^2$$

When $\sqrt{\frac{P}{P_E}} = (3-\varepsilon)$, $\tan\frac{\pi}{2}(3-\varepsilon) - \frac{1}{3}\tan\frac{\pi}{6}(3-\varepsilon)$

$$= \tan\left(\frac{\pi}{2} - \frac{\pi\varepsilon}{2}\right) - \frac{1}{3}\tan\left(\frac{\pi}{2} - \frac{\pi\varepsilon}{6}\right)$$

$$= \cot\frac{\pi\varepsilon}{2} - \frac{1}{3}\cot\frac{\pi\varepsilon}{6} = \frac{\left[1 - \frac{1}{2!}\left(\frac{\pi\varepsilon}{2}\right)^2 + \cdots\right]}{\frac{\pi\varepsilon}{2}\left[1 - \frac{1}{3!}\left(\frac{\pi\varepsilon}{2}\right)^2 + \cdots\right]} - \frac{\left[1 - \frac{1}{2!}\left(\frac{\pi\varepsilon}{6}\right)^2 + \cdots\right]}{\frac{\pi\varepsilon}{2}\left[1 - \frac{1}{3!}\left(\frac{\pi\varepsilon}{6}\right)^2 + \cdots\right]}$$

$$= \frac{1}{\left(\frac{\pi\varepsilon}{2}\right)}\left[\left(1 - \frac{1}{3}\cdot\left(\frac{\pi\varepsilon}{2}\right)^2 + \cdots\right) - \left(1 - \frac{1}{3}\left(\frac{\pi\varepsilon}{6}\right)^2 + \cdots\right)\right]$$

$$= -\frac{8}{27}\left(\frac{\pi\varepsilon}{2}\right) + \cdots\cdots \quad ; \quad \tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\tan^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}} = -\frac{16}{27}$$

$$\alpha = \frac{1}{2} - \frac{1}{9} = \frac{7}{18}$$

$$\beta = \frac{1}{9} \quad ; \quad \frac{\varepsilon}{\ell} = \left(\frac{\delta}{\ell}\right)^2\frac{7}{2} = 3.5\left(\frac{\delta}{\ell}\right)^2$$

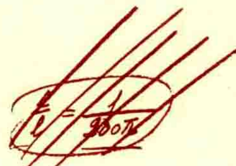
$\frac{9}{8\pi} = 0.35810$, $\frac{3}{4\pi} = 0.23$ [illegible]3 <u>4</u>

①	②	③	④	⑤	⑥	⑦	⑧	⑨
$\sqrt{\frac{P}{P_E}}$	$\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$	$\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}$	② $-\frac{1}{3}$③	② $+\frac{1}{3}$③	④ × ⑤	α	β	α/β
3.0								3.5000
2.9	6.3138	19.081	−0.04653	12.67413	−0.58973	0.39517	0.11368	3.4762
2.8	3.0777	9.5144	−0.09377	6.24917	−0.58598	0.40212	0.11651	3.4514
2.7	1.9626	6.3138	−0.14200	4.06720	−0.57754	0.41054	0.11963	3.4317
2.6	1.3764	4.7046	−0.19180	2.94460	−0.56477	0.42052	0.12316	3.4144
2.5	1.0000	3.7321	−0.24403	2.24403	−0.54761	0.43228	0.12718	3.3990
2.4	0.72654	3.0777	−0.29936	1.75244	−0.52461	0.44630	0.13185	3.3849
2.3	0.50953	2.6051	−0.35884	1.37790	−0.49445	0.46316	0.13733	3.3726
2.2	0.32492	2.2460	−0.42375	1.07359	−0.45493	0.48367	0.14388	3.3616
2.1	0.15838	1.9626	−0.49582	0.81258	−0.40289	0.50901	0.15186	3.3518
2.0	0	1.7321	−0.57737	0.57737	−0.33336	0.54087	0.16181	3.3426
1.9	−0.15838	1.5399	−0.67168	0.35492	−0.23839	0.58190	0.17450	3.3347
1.8	−0.32492	1.3764	−0.78372	0.13388	−0.10492	0.13624	0.19120	3.3276
1.7	−0.50953	1.2349	−0.92116	−0.09790	+0.09018	0.71094	0.21408	3.3209
1.6	−0.72654	1.1106	−1.09674	−0.35634	+0.39081	0.81874	0.24698	3.3150
1.5	−1.00000	1.0000	−1.33333	−0.66667	+0.88889	0.98498	0.29760	3.3097
1.4	−1.3764	0.90040	−1.67653	−1.07626	+1.80438	1.26715	0.38342	3.3049
1.3	−1.9626	0.80978	−2.23253	−1.69267	+3.77894	1.82353	0.55251	3.3004
1.2	−3.0777	0.72654	−3.31987	−2.83552	+9.41356	3.25575	0.98762	3.2966
1.1	−6.3138	0.64941	−6.53027	−6.09733	+39.81656	10.09151	3.06455	3.2930
1.0	−∞	0.57735						3.2899
0.9	+6.3138	0.50953	+6.14396	+6.48364	+39.83522	5.52449	1.68063	3.2872
0.8	+3.0777	0.44523	+2.92929	+3.22611	+9.45021	0.96069	0.29248	3.2846

$$\frac{\varepsilon_{comp}}{l} = \frac{P}{AE} = \left(\frac{P}{P_E}\right)\left(\frac{P_E}{AE}\right)$$

$$P_E = \frac{\pi^2 EI}{l^2}, \quad \frac{P_E}{AE} = \frac{\pi^2 I}{Al^2} = \pi^2\left(\frac{r}{l}\right)^2$$

$$\frac{\varepsilon_{comp}}{l} = \left(\frac{P}{P_E}\right)\pi^2\left(\frac{r}{l}\right)^2$$

Let 

$$\frac{\varepsilon_{Tot.}}{l} = \left(\frac{P}{P_E}\right)\pi^2\left(\frac{r}{l}\right)^2 + \left(\frac{\delta}{l}\right)^2 \cdot \frac{d}{\beta}$$

$$\frac{r}{l} = \frac{1}{100\pi}$$

$$\frac{\varepsilon_{Tot}}{l} = \left(\frac{r}{l}\right)^2\left[\left(\frac{P}{P_E}\right)\pi^2 + \left(\frac{\delta}{r}\right)^2 \cdot \frac{d}{\beta}\right] = \frac{\varepsilon}{l} + \left(\frac{\pi}{500}\right)^2 \frac{P}{P_E}$$

$$= \frac{\varepsilon}{l} +$$

$$= \left(\frac{P}{P_E}\right)\left(\frac{1}{100}\right)^2 + \left(\frac{\delta}{l}\right)^2 \cdot \frac{d}{\beta} \rightarrow \frac{\varepsilon}{l}$$

$\delta^* =$ new delta $= \frac{1}{10}\delta$

$$= \left(\frac{P}{P_E}\right)\left(\frac{1}{100}\right)^2 + \left(\frac{1}{10}\right)^2 \frac{2}{\beta}$$

$$\frac{\varepsilon_{Tot}}{l} \Big/ \pi^2\left(\frac{r}{l}\right)^2 = \left(\frac{P}{P_E}\right) + 400\left(\frac{\varepsilon}{l}\right)_{old.}$$

!!! Has To be Corrected !!!

$\sqrt{\frac{P}{P_E}}$	$\left(\frac{\varepsilon}{l}\right)_1$	$\left(\frac{\varepsilon}{l}\right)_2$	$\frac{\varepsilon_{TOT}}{l} / \left(\frac{R}{l}\right)^2_1$	$\frac{\varepsilon_{TOT}}{l} / \left(\frac{R}{l}\right)^2_2$	$\frac{P}{P_E}$	
3.0	0	0.140000	9.0000	~~65.000~~ 23.000	9.00	
2.9	0.000049	0.133928	8.4296	61.9812	8.41	
2.8	0.000190	0.127999	7.9160	59.0396	7.84	
2.7	0.000426	0.122429	7.4604	56.2616	7.29	
2.6	0.000751	0.117090	7.0604	53.5960	6.76	
2.5	0.001166	0.111956	6.7164	51.0324	6.25	
2.4	0.001669	0.106990	6.4246	48.5560	5.76	
2.3	0.002266	0.102200	6.1964	46.1700	5.29	
2.2	0.002955	0.097564	6.0220	43.8656	4.84	
2.1	0.003734	0.093056	5.9036	41.6324	4.41	
2.0	0.004609	0.088659	5.8436	39.4636	4.00	
1.9	0.005586	0.084385	5.8444	37.3640	3.61	
1.8	0.006662	0.080208	5.9048	35.3232	3.24	
1.7	0.008073	0.075408	6.3192	33.0532	2.89	
1.6	0.009153	0.072078	6.2212	31.3912	2.56	
1.5	0.010584	0.068107	6.4736	29.4928	2.25	
1.4	0.012159	0.064168	6.8236	27.6272	1.96	
1.3	0.013901	0.060239	7.2504	25.7856	1.69	
1.2	0.015853	0.051270	7.7812	23.9480	1.44	
1.1	0.018085	0.052194	8.4440	22.6876	1.21	
1.0	0.020726	0.047871	9.2704	20.1484	1.00	
0.9	0.024154	0.042931	10.4716	17.9824	0.81	
0.8	0.031256	0.034475	13.1424	14.4300	0.64	

$$\frac{F}{P_E} = \xi\left[27.00 - 5640\xi + 282000\xi^2\right]$$

$$\xi_1 = A\left[27.00\xi_1 - 5640\xi_1^2 + 282000\xi_1^3\right] + B\left[27.000\xi_2 - 5640\xi_2^2 + 282000\xi_2^3\right]$$

$$\xi_2 = B\left[27.00\xi_1 - 5640\xi_1^2 + 282000\xi_1^3\right] + A\left[27.000\xi_2 - 5640\xi_2^2 + 282000\xi_2^3\right]$$

$$\frac{\xi_1}{B} - \frac{\xi_2}{A} = \left(\frac{A}{B} - \frac{B}{A}\right)\left[27.00\xi_1 - 5640\xi_1^2 + 282000\xi_1^3\right]$$

$$\frac{\xi_2}{B} - \frac{\xi_1}{A} = \left(\frac{A}{B} - \frac{B}{A}\right)\left[27.00\xi_2 - 5640\xi_2^2 + 282000\xi_2^3\right]$$

$$\frac{\xi_2}{A} = \frac{\xi_1}{B} - \left(\frac{A}{B} - \frac{B}{A}\right)\left[27.00\xi_1 - 5640\xi_1^2 + 282000\xi_1^3\right]$$

$$\frac{\xi_2}{A} = \frac{\xi_1}{B} - \left(\frac{A}{B} - \frac{B}{A}\right)\left(\frac{F_1}{P_E}\right)$$

From the symmetry of the equations, $\xi_1 = \xi_2$ ∴ the solution can be only symmetrical ! !!! wrong !!!

A more direct proof of summation:

Let $$S = \sum_{n=1,2,3}^{\infty} \frac{1}{x^2-n^2}$$

We have $$\sin \pi x = \pi x \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$$

$$\therefore \log \sin \pi x = \log \pi x + \sum_{n=1}^{\infty} \log\left(1-\frac{x^2}{n^2}\right)$$

Differentiating with respect to x,

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{-\frac{2x}{n^2}}{1-\frac{x^2}{n^2}} = \frac{1}{x} - 2x\sum_{n=1}^{\infty}\frac{1}{n^2-x^2}$$

$$= \frac{1}{x} + 2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{x^2-n^2} = \frac{\pi}{2x}\cot \pi x - \frac{1}{2x^2}$$

Hence $$\sum_{n=1,3,5}^{\infty} \frac{1}{\frac{P}{P_E}-n^2} = \sum_{n=1,2,3}^{\infty} \frac{1}{\frac{P}{P_E}-n^2} - \frac{1}{4}\sum_{n=1,2,3}^{\infty}\frac{1}{\frac{P}{4P_E}-n^2}$$

$$= \frac{\pi}{2\sqrt{\frac{P}{P_E}}}\cot \pi\sqrt{\frac{P}{P_E}} - \frac{1}{2\frac{P}{P_E}} - \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\cot\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{1}{2\frac{P}{P_E}}$$

$$\sum_{n=1,3,5}^{\infty} = \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\left\{2\cot \pi\sqrt{\frac{P}{P_E}} - \cot\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\} = -\frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$A = \frac{3}{2\pi^2}\left[\frac{1}{\left(\frac{P}{P_E}\right)}\left\{\frac{\pi^2}{6} + \frac{\pi}{2\sqrt{\frac{P}{P_E}}}\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{2\frac{P}{P_E}}\right\}\right.$$

$$\left. - \frac{1}{\frac{P}{P_E}}\left\{\frac{\pi^2}{6}\frac{1}{9} + \frac{\pi}{6\sqrt{\frac{P}{P_E}}}\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}} - \frac{1}{2\frac{P}{P_E}}\right\}\right]$$

$$A = \frac{1}{\frac{P}{P_E}}\left[\frac{2}{9} + \frac{1}{4\pi\sqrt{\frac{P}{P_E}}}\left\{3\cot\pi\sqrt{\frac{P}{P_E}} - \cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right\}\right] \quad \text{O.K.}$$

~~$$A = \frac{3}{\pi^2}\left[\frac{1}{\frac{P}{P_E}}\left\{\frac{\pi^2}{8} + \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\left(2\cot\pi\sqrt{\frac{P}{P_E}} - \cot\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)\right\}\right.$$~~

~~$$- \frac{1}{\frac{P}{P_E}}\left\{\frac{\pi^2}{8}\frac{1}{9} + \frac{\pi}{}\right.$$~~

$$\sum_{n=1,3,5}^{\infty}\frac{1}{\frac{P}{P_E} - n^2} = \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\left\{2\cot\pi\sqrt{\frac{P}{P_E}} - \cot\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}$$

$$2\cot 2\theta - \cot\theta = \frac{2\cos 2\theta}{\sin 2\theta} - \frac{\cos\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta}\left[\frac{\cos 2\theta}{\cos\theta} - \cos\theta\right] = \frac{1}{\sin\theta}\,\frac{\cos^2\theta - \sin^2\theta - \cos^2\theta}{\cos\theta}$$

$$= -\tan\theta$$

$$\therefore \sum_{n=1,3,5}^{\infty}\frac{1}{\frac{P}{P_E} - n^2} = -\frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{\frac{P}{P_E} - n^2} = -\frac{\pi}{4\sqrt{\frac{P}{P_E}}} \tan \frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$\sum_{n=1,2,3,}^{\infty} \frac{1}{\frac{P}{P_E} - n^2} = \frac{\pi}{2\sqrt{\frac{P}{P_E}}} \cot \pi\sqrt{\frac{P}{P_E}} - \frac{1}{2\left(\frac{P}{P_E}\right)}$$

Section 3

Buckling of Column with Three Non-linear Suppertes

Three Supports ! Symmetrical

$$\frac{l}{4}\left(\frac{\pi}{l}\right)^2 \sum_{n=1,3,5}^{\infty} n^2\left[n^2 P_E - P\right] a_n^2 + 2W_1 + W_3$$

$$\frac{l}{2}\left(\frac{\pi}{l}\right)^2 n^2\left[n^2 - \frac{P}{P_E}\right] a_n + 2\sin\frac{n\pi}{4}\left(\frac{F_1}{P_E}\right) + \sin\frac{n\pi}{2}\left(\frac{F_2}{P_E}\right) = 0$$

$$\frac{a_n}{l} = \frac{2}{\pi^2}\,\frac{2\sin\frac{n\pi}{4}\cdot\left(\frac{F_1}{P_E}\right) + \sin\frac{n\pi}{2}\cdot\left(\frac{F_2}{P_E}\right)}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

$$\frac{\delta_1}{l} = \frac{2}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{2\sin^2\frac{n\pi}{4}\left(\frac{F_1}{P_E}\right) + \sin\frac{n\pi}{2}\sin\frac{n\pi}{4}\left(\frac{F_2}{P_E}\right)}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

$$\frac{\delta_2}{l} = \frac{2}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{2\sin\frac{n\pi}{4}\sin\frac{n\pi}{2}\left(\frac{F_1}{P_E}\right) + \sin^2\frac{n\pi}{2}\left(\frac{F_2}{P_E}\right)}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

$$\frac{\delta_1}{l} = \frac{2}{\pi^2}\left[\left(\frac{F_1}{P_E}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]} + \left(\frac{F_2}{P_E}\right)\frac{1}{\sqrt{2}}\left\{\sum_{n=1,5,9}^{\infty}(-1)^{\frac{n-1}{4}}\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]} + \sum_{3,7,11}^{\infty}(-1)^{\frac{n+1}{4}}\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]}\right\}\right]$$

$$\left.\begin{aligned}\frac{\delta_1}{l} &= \alpha\frac{F_1}{P_E} + \beta\frac{F_2}{P_E} \\ \frac{\delta_2}{l} &= 2\beta\frac{F_1}{P_E} + \alpha\frac{F_2}{P_E}\end{aligned}\right\}$$

$$\alpha = \frac{2}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]}$$

$$\beta = \frac{1}{\sqrt{2}\,\pi^2}\left\{\sum_{n=1,5,9}^{\infty}(-1)^{\frac{n-1}{4}}\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]} + \sum_{n=3,7,11}^{\infty}(-1)^{\frac{n+1}{4}}\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]}\right\}$$

$$\therefore\ \alpha = \frac{2}{\pi^2\frac{P}{P_E}}\left\{\sum_{n=1,3,5}^{\infty}\frac{1}{n^2} + \sum_{n=1,3,5}^{\infty}\frac{1}{\frac{P}{P_E}-n^2}\right\}$$

$$\alpha = \frac{2}{\pi^2\frac{P}{P_E}}\left\{\frac{\pi^2}{8} - \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}$$

$$\boxed{\alpha = \frac{1}{\frac{P}{P_E}}\left\{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}$$

$$\sum_{n=1,5,9,13}^{\infty} (-1)^{\frac{n-1}{4}} \frac{1}{n^2\left[\frac{P}{P_E} - n^2\right]} = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(1+4m)^2\left[\frac{P}{P_E} - (1+4m)^2\right]}$$

$$= \sum_{m=0}^{\infty} \frac{1}{(1+4m)^2\left[\frac{P}{P_E} - (1+4m)^2\right]} - 2\sum_{m=0}^{\infty} \frac{1}{(5+8m)^2\left[\frac{P}{P_E} - (5+8m)^2\right]}$$

Investigate $\displaystyle\sum_{m=0}^{\infty} \frac{1}{\frac{P}{P_E} - (1+4m)^2} = -\frac{1}{2\sqrt{\frac{P}{P_E}}} \sum \left(\frac{1}{(1+4m) - \sqrt{\frac{P}{P_E}}} - \frac{1}{(1+4m) + \sqrt{\frac{P}{P_E}}} \right)$

$$\sum \frac{1}{(1+4m) - \sqrt{\frac{P}{P_E}}} = \sum \int_0^{\infty} e^{-x\left[(1+4m) - \sqrt{\frac{P}{P_E}}\right]} dx$$

$$= \int_0^{\infty} e^{-x\left(1-\sqrt{\frac{P}{P_E}}\right)} \sum \left(e^{-4x}\right)^m dx = \int_0^{\infty} e^{-x\left(1-\sqrt{\frac{P}{P_E}}\right)} \frac{dx}{1-e^{-4x}}$$

$$= \frac{1}{4} \int_0^{\infty} \frac{e^{-\xi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{4}\right)} d\xi}{1-e^{-\xi}}$$

$$\boxed{\sum_{m=0}^{\infty} \frac{1}{\frac{P}{P_E} - (1+4m)^2} = \frac{1}{8\sqrt{\frac{P}{P_E}}} \left\{ \psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{4}\right) - \psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{4}\right) \right\}}$$

$$\sum_{m=0}^{\infty} \frac{1}{(1+4m)^2} = -\frac{1}{32} \lim_{4\sqrt{\frac{P}{P_E}} \to 0} \frac{-\left\{\psi\left(\frac{1}{4}\right) - \psi\left(\frac{1}{4} - \frac{\sqrt{\frac{P}{P_E}}}{4}\right)\right\} - \left\{\psi\left(\frac{1}{4} + \frac{\sqrt{\frac{P}{P_E}}}{4}\right) - \psi\left(\frac{1}{4}\right)\right\}}{\frac{1}{4}\sqrt{\frac{P}{P_E}}}$$

$$= + \frac{1}{16} \psi'\left(\frac{1}{4}\right)$$

$$\sim \quad \boxed{\sum_{m=0}^{\infty} \frac{1}{(1+4m)^2} = \frac{1}{16} \sum_{m=0}^{\infty} \frac{1}{\left(\frac{1}{4}+m\right)^2} = + \frac{1}{16} \psi'\left(\frac{1}{4}\right)}$$

$$\therefore \boxed{\sum_{m=0}^{\infty} \frac{1}{(1+4m)^2\left[\frac{P}{P_E} - (1+4m)^2\right]} = \frac{1}{\frac{P}{P_E}}\left[\frac{1}{16}\psi'\left(\frac{1}{4}\right) + \frac{1}{8\sqrt{\frac{P}{P_E}}}\left\{\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{4}\right) - \psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{4}\right)\right\}\right]}$$

$$\sum_{m=0}^{\infty} \frac{1}{\frac{P}{P_E} - (5+8m)^2} = \sum \frac{1}{2\sqrt{\frac{P}{P_E}}}\left\{\frac{1}{(5+8m) - \sqrt{\frac{P}{P_E}}} - \frac{1}{(5+8m) + \sqrt{\frac{P}{P_E}}}\right\}$$

$$= \frac{1}{16\sqrt{\frac{P}{P_E}}}\left\{\psi\left(\frac{5-\sqrt{\frac{P}{P_E}}}{8}\right) - \psi\left(\frac{5+\sqrt{\frac{P}{P_E}}}{8}\right)\right\}$$

$$\boxed{\sum_{m=0}^{\infty} \frac{1}{(5+8m)^2\left[\frac{P}{P_E} - (5+8m)^2\right]} = \frac{1}{\frac{P}{P_E}}\left[\frac{1}{64}\psi'\left(\frac{5}{8}\right) + \frac{1}{16\sqrt{\frac{P}{P_4}}}\left\{\psi\left(\frac{5-\sqrt{\frac{P}{P_E}}}{8}\right) - \psi\left(\frac{5+\sqrt{\frac{P}{P_E}}}{8}\right)\right\}\right]}$$

$$\sum_{n=1,5,9,13}^{\infty} (-)^{\frac{n-1}{4}} \frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]} = \frac{1}{16\frac{P}{P_E}}\left[\left\{\psi'\left(\tfrac{1}{4}\right)-\tfrac{1}{2}\psi'\left(\tfrac{5}{8}\right)\right\}+\frac{2}{\sqrt{\frac{P}{P_E}}}\left\{\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{4}\right)-\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{4}\right)\right.\right.$$
$$\left.\left.-\psi\left(\frac{5-\sqrt{\frac{P}{P_E}}}{8}\right)+\psi\left(\frac{5+\sqrt{\frac{P}{P_E}}}{8}\right)\right\}\right]$$

$$\sum_{n=3,7,11,}^{\infty} (-1)^{\frac{n+1}{4}} \frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]} = -\sum_{m=0}^{\infty}(-)^m \frac{1}{(4m+3)^2\left[\frac{P}{P_E}-(4m+3)\right]^2}$$

$$= -\sum_{m=0}^{\infty}\frac{1}{(4m+3)^2\left[\frac{P}{P_E}-(4m+3)^2\right]}+2\sum_{m=0}^{\infty}\frac{1}{(8m+7)^2\left[\frac{P}{P_E}-(8m+7)^2\right]^2}$$

$$= -\frac{1}{16\frac{P}{P_E}}\left[\left\{\psi'\left(\tfrac{3}{4}\right)-\tfrac{1}{2}\psi'\left(\tfrac{7}{8}\right)\right\}+\frac{2}{\sqrt{\frac{P}{P_E}}}\left\{\psi\left(\frac{3-\sqrt{\frac{P}{P_E}}}{4}\right)-\psi\left(\frac{3+\sqrt{\frac{P}{P_E}}}{4}\right)\right.\right.$$
$$\left.\left.-\psi\left(\frac{7-\sqrt{\frac{P}{P_E}}}{8}\right)+\psi\left(\frac{7+\sqrt{\frac{P}{P_E}}}{8}\right)\right\}\right]$$

$$\beta = \frac{1}{16\sqrt{2}\pi^2\frac{P}{P_E}}\left[\left\{\psi'\left(\tfrac{1}{4}\right)-\psi'\left(\tfrac{3}{4}\right)+\tfrac{1}{2}\psi'\left(\tfrac{7}{8}\right)-\tfrac{1}{2}\psi'\left(\tfrac{5}{8}\right)\right\}\right.$$
$$+\frac{2}{\sqrt{\frac{P}{P_E}}}\left\{\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{4}\right)+\psi\left(\frac{3+\sqrt{\frac{P}{P_E}}}{4}\right)-\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{4}\right)-\psi\left(\frac{3-\sqrt{\frac{P}{P_E}}}{4}\right)\right.$$
$$\left.\left.+\psi\left(\frac{7-\sqrt{\frac{P}{P_E}}}{8}\right)+\psi\left(\frac{5+\sqrt{\frac{P}{P_E}}}{8}\right)-\psi\left(\frac{7+\sqrt{\frac{P}{P_E}}}{8}\right)-\psi\left(\frac{5-\sqrt{\frac{P}{P_E}}}{8}\right)\right\}\right]$$

$$\psi'(\tfrac{1}{4}) = 17.197329$$
$$\psi'(\tfrac{3}{4}) = 2.541880$$
$$14.655449$$

$$\frac{0.005}{1!} = 0.005$$

$$\frac{0.005^2}{2!} = 0.00001250$$

$$\frac{0.005^3}{3!} = 0.0000000208$$

$$\psi'(x) = \psi'(1+x) + \frac{1}{x^2}$$

$$\psi'(\tfrac{5}{8}) = 0.841183 + 2.56$$
$$\psi'(\tfrac{7}{8}) = 0.699619 + 1.306122$$
$$0.141564 + 1.253878$$

$$\tfrac{1}{2} \times (\qquad) =$$

$$\therefore \boxed{\psi'(\tfrac{1}{4}) - \psi'(\tfrac{3}{4}) + \tfrac{1}{2}\psi'(\tfrac{7}{8}) - \tfrac{1}{2}\psi'(\tfrac{5}{8}) = 13.957728}$$

$$\sqrt{\frac{P}{P_E}} = 4 \, ; \quad \frac{P}{P_E} = 16$$

$$\therefore \quad \alpha = 0.0156250$$

$$\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{4}\right) = \psi(-0.75) = -2.894120 \quad ; \quad \psi\left(\frac{3+\sqrt{\frac{P}{P_E}}}{4}\right) = \psi(1.75) = 0.247472$$

$$\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{4}\right) = \psi(1.25) = -0.227454 \quad ; \quad \psi\left(\frac{3-\sqrt{\frac{P}{P_E}}}{4}\right) = \psi(-0.25) = +2.914139$$

$$\psi\left(\frac{7-\sqrt{\frac{P}{P_E}}}{8}\right) = \psi(0.375) = -2.753999 \quad ; \quad \psi\left(\frac{5+\sqrt{\frac{P}{P_E}}}{8}\right) = \psi(1.125) = -0.388493$$

$$\psi\left(\frac{7+\sqrt{\frac{P}{P_E}}}{8}\right) = \psi(1.375) = -0.087332 \quad ; \quad \psi\left(\frac{5-\sqrt{\frac{P}{P_E}}}{8}\right) = \psi(0.125) = -8.388493$$

$$\beta = 0.0039062 \qquad \sqrt{2}\,\beta = 0.00552427$$

Let $\frac{F_1}{P_E} = a\xi_1 - b\xi_1^2 + c\xi_1^3$

$$\frac{F_2}{P_E} = a\xi_2 - b\xi_2^2 + c\xi_2^3$$

for $\xi_1 = \xi_2 \to 0$;

$$\xi_1 = [\alpha\xi_1 + \beta\xi_2]a \quad \text{or} \quad (a\alpha - 1)\xi_1 + a\beta\xi_2 = 0$$

$$\xi_2 = [2\beta\xi_1 + \alpha\xi_2]a \quad\quad (2a\beta)\xi_1 + (a\alpha - 1)\xi_2 = 0$$

$$\therefore \quad a^2\alpha^2 - 2a\alpha + 1 - 2a^2\beta^2 = 0$$

$$(\alpha^2 - 2\beta^2)a^2 - (2\alpha)a + 1 = 0$$

$$a = \frac{\alpha}{\alpha^2 - 2\beta^2} \pm \sqrt{\frac{\alpha^2}{(\alpha^2 - 2\beta^2)^2} - \frac{1}{\alpha^2 - 2\beta^2}}$$

$$\boxed{a = \frac{\alpha \pm \sqrt{2}\beta}{\alpha^2 - 2\beta^2}} \quad\quad \boxed{a = \frac{1}{\alpha \pm \sqrt{2}\beta}}$$

for $\sqrt{\frac{P}{P_E}} = 4$; $\quad a = \underline{47.2829}$ or $\underline{99.0030}$

$$\frac{F_1}{P_E} = \xi_1 (47.2829 - 9876.88\xi_1 + 493844\,\xi_1^2)$$

$$\frac{F_2}{P_E} = \xi_2 (47.2829 - 9876.88\,\xi_2 + 493844\,\xi_2^2)$$

$\sqrt{\frac{P}{P_E}} = 1.6$; $\frac{P}{P_E} = 2.56$

$\psi\left(\frac{1-\sqrt{\frac{P}{P_E}}}{4}\right) = \psi(-0.15) = 5.811396$; $\psi\left(\frac{3+\sqrt{\frac{P}{P_E}}}{4}\right) = \psi(1.1500) = -0.354327$

$\psi\left(\frac{1+\sqrt{\frac{P}{P_E}}}{4}\right) = \psi(0.65) = -1.370349$; $\psi\left(\frac{3-\sqrt{\frac{P}{P_E}}}{4}\right) = \psi(0.35) = -2.971071$

$\psi\left(\frac{7-\sqrt{\frac{P}{P_E}}}{8}\right) = \psi(0.675) = -1.292955$; $\psi\left(\frac{5+\sqrt{\frac{P}{P_E}}}{8}\right) = \psi(0.825) = -0.908867$

$\psi\left(\frac{7+\sqrt{\frac{P}{P_E}}}{8}\right) = \psi(1.075) = -0.460181$; $\psi\left(\frac{5-\sqrt{\frac{P}{P_E}}}{8}\right) = \psi(0.425) = -2.388996$

$\beta = 0.047253$

$\alpha = 0.125887$

$$\xi_1 = 0.125887\,\xi_1\left(47.2829 - 9876.88\,\xi_1 + 493844\,\xi_1^2\right)$$
$$+ 0.047253\,\xi_2\left(47.2829 - 1876.88\,\xi_2 + 493844\,\xi_2^2\right)$$

$$\xi_2 = 0.094506\,\xi_1\left(47.2829 - 9876.88\,\xi_1 + 493844\,\xi_1^2\right)$$
$$+ 0.125887\,\xi_2\left(47.2829 - 9876.88\,\xi_1 + 493844\,\xi_2^2\right)$$

$$21.1627\,\xi_1 = 2.66411\,\xi_1\,(47.2829 - 9876.88\,\xi_1 + 493844\,\xi_1^2) + \xi_2\,(\qquad)$$

$$7.94363\,\xi_2 = 0.75072\,\xi_1\,(\qquad) + \xi_2\,(\qquad)$$

$$\therefore\ 7.94363\,\xi_2 = 21.1627\,\xi_1 - 1.91339\,\xi_1\,(\qquad)$$

$$\boxed{\xi_2 = -8.72497\,\xi_1 + 2379.05\,\xi_1^2 - 118953\,\xi_1^3}$$

$$7.94363\,\xi_1 = \xi_1\,(\qquad) + 0.375356\,\xi_2\,(\qquad)$$

$$10.5813\,\xi_2 = \xi_1\,(\qquad) + 1.332048\,\xi_2\,(\qquad)$$

$$\boxed{\xi_1 = -3.52429\,\xi_2 + 893.003\,\xi_2^2 - 44650.2\,\xi_2^3}$$

$\eta_1 = 100\xi_1$

$\eta_2 = 100\xi_2$

or $\xi_1 = \dfrac{\eta_1}{100}$

$\xi_2 = \dfrac{\eta_2}{100}$

$$\eta_2 = -\left[8.72497 - 23.7905\,\eta_1 + 11.8953\,\eta_1^2\right]\eta_1$$

$$\eta_1 = -\left[3.52429 - 8.93003\,\eta_2 + 4.46502\,\eta_2^2\right]\eta_2$$

Put $\eta_2/\eta_1 = \rho$

$$\rho = -\left[8.72497 - 23.7905\,\eta_1 + 11.8953\,\eta_1^2\right]$$

$$1 = -\left[3.52429\,\rho - 8.93003\,\rho^2\eta_1 + 4.46502\,\rho^3\eta_1^2\right]$$

$$\therefore \quad \rho^2 = 76.12510 - 415.1428\,\eta_1 + 773.5602\,\eta_1^2 - 565.9879\,\eta_1^3 + 141.4970\,\eta_1^4$$

$$8.72497 - 23.7905\,\eta_1 + 11.8953\,\eta_1^2$$

$$-\rho^3 = 664.18921 - 3622.1085\,\eta_1 + 6749.2895\,\eta_1^2 - 4938.2274\,\eta_1^3 + 1234.5571\,\eta_1^4$$
$$- 1811.0542\,\eta_1 + 9876.4548\,\eta_1^2 - 18403.384\,\eta_1^3 + 13465.1351\,\eta_1^4 - 3366.2844\,\eta_1^5$$
$$+ 905.5271\,\eta_1^2 - 4938.2274\,\eta_1^3 + 9201.692\,\eta_1^4 - 6732.5676\,\eta_1^5$$
$$+ 1683.1422\,\eta_1^6$$

$$-\rho^3 = +664.18921 - 5433.1627\,\eta_1 + 17531.2714\,\eta_1^2 - 28279.8388\,\eta_1^3 + 23901.3842\,\eta_1^4 - 10098.8520\,\eta_1^5 + 1683.1422\,\eta_1^6$$

$$1 = 30.7493 - 83.8446\,\eta_1 + 41.9223\,\eta_1^2$$
$$679.7994 - 3707.2377\,\eta_1 + 6907.9158\,\eta_1^2 - 5054.2889\,\eta_1^3 + 1263.5725\,\eta_1^4$$
$$2965.6181 - 24259.1801\,\eta_1 + 78277.4774\,\eta_1^2 - 126270.0458\,\eta_1^3 + 106720.1585\,\eta_1^4 - 45091.5762\,\eta_1^5 + 7515.2636\,\eta_1^6$$

$$F(\eta_1) = \eta_1^8 - 6.00000\,\eta_1^7 + 14.2005\,\eta_1^6 - 16.6337\,\eta_1^5 + 9.74326\,\eta_1^4 - 2.30880\,\eta_1^3 - 0.093103\,\eta_1^2 + 0.079299\,\eta_1 + 0.0039585 = 0$$

$$F'(\eta_1) = 8\eta_1^7 - 42.0000\,\eta_1^6 + 85.2030\,\eta_1^5 - 83.1685\,\eta_1^4 + 38.9730\,\eta_1^3 - 6.92640\,\eta_1^2 - 0.186206\,\eta_1 + 0.079299$$ ✓

$$\eta_1^2 - 0.151173\,\eta_1 - 0.042517 = 0$$

$$\eta_1 \approx 0.42587 \pm \sqrt{0.42587^2 + 0.042517} = - 0.0472$$

$F_1(-0.0490) = 0.0001821$ $\quad F'(-0.0490) = +0.06670$

$F_1(-0.05173) = 0.0000030$ $\quad F'(-0.05173) = +0.06437$

$$\boxed{\eta_1 = -0.051777}$$

$$F(\eta_1) = \eta_1^7 - 6.051777\,\eta_1^6 + 14.3183\,\eta_1^5 - 17.3751\,\eta_1^4 + 10.6429\,\eta_1^3 - 2.85986\,\eta_1^2 + 0.054972\,\eta_1 - 0.076453 = 0$$

$$F'(\eta_1) = 7\eta_1^6 - 36.3107\,\eta_1^5 + 71.5915\,\eta_1^4 - 69.5004\,\eta_1^3 + 31.9287\,\eta_1^2 - 5.71972\,\eta_1 + 0.054972$$

To find the negative roots

$$F(\eta_1) = \eta_1^8 + 6.0000\eta_1^7 + 14.2005\eta_1^6 + 16.6337\eta_1^5 + 9.74326\eta_1^4 + 2.30880\eta_1^3 - 0.093103\eta^2 - 0.079299\eta + 0.0039585 = 0$$

$$F'(-\eta_1) = 8\eta_1^7 + 42.\eta_1^6 + 85.2030\eta^5 + 83.1685\eta_1^4 + 38.9730\eta_1^3 + 6.92640\eta_1^2 - 0.186206\eta_1 - 0.079299$$

$F(-0.0490) = 0.000183$; $F'(-\eta_1) = -0.06670$

$F(-0.051777) =$

$\eta_1 = -0.051777$

$$F(-\eta_1) = \eta_1^7 + 6.051777\eta_1^6 + 14.5138\eta_1^5 + 17.3852\eta_1^4 + 10.6434\eta_1^3 + 2.159884\eta_1^2 + 0.054973\eta_1 - 0.076453 = 0$$

$$F'(-\eta_1) = 7\eta_1^6 + 36.3107\eta_1^5 + 71.5915\eta_1^4 + 69.5004\eta_1^3 + 31.9287\eta_1^2 + 5.71972\eta_1 + 0.054973$$

$F(-0.125) = +0.000603 \quad ; \qquad F'(-0.125) = +1.423$

$$\boxed{\eta_1 = -0.124576}$$

$F(\eta_1) = \eta_1^6 - 6.17635\,\eta_1^5 + 15.2832\,\eta_1^4 - 19.2891\,\eta_1^3 + 13.04636\,\eta_1^2 - 4.48515\,\eta_1 + 0.613715 = 0$

$F'(\eta_1) = 6\eta_1^5 - 30.8818\,\eta_1^4 + 61.1328\,\eta_1^3 - 57.8673\,\eta_1^2 + 26.09272\,\eta_1 - 4.48515$

$F(0.69) = -0.000130 \quad ; \qquad F'(0.69) = -0.010716$

$F(0.67786) = -0.000017 ; \qquad F'(0.67786) = -0.007933$

$F(0.675718) = \text{o.k.} \qquad \boxed{\eta_1 = 0.675718}$

$F(\eta_1) = \eta_1^5 - 5.50063\,\eta_1^4 + 11.5663\,\eta_1^3 - 11.4735\,\eta_1^2 + 5.29349\,\eta_1 - 0.908243 = 0$

$F'(\eta_1) = 5\eta_1^4 - 22.0025\,\eta_1^3 + 34.6989\,\eta_1^2 - 22.9470\,\eta_1 + 5.29349$

$F(0.466) = +0.000015 \quad ; \qquad F'(0.466) = 0.1445$

$$\boxed{\eta_1 = 0.465896}$$

$$F(\eta_1) = \eta_1^4 - 5.03473\,\eta_1^3 + 9.22064\,\eta_1^2 - 7.17764\,\eta_1 + 1.94946 = 0$$

$$F'(\eta_1) = 4\eta_1^3 - 15.10419\,\eta_1^2 + 18.44128\,\eta_1 - 7.17764$$

$F(0.607) = -0.000287$; $F'(0.607) = -0.6543$

$F(0.606561) =$ ок. $\boxed{\eta_1 = 0.606561}$

$$F(\eta_1) = \eta_1^3 - 4.42817\,\eta_1^2 + 6.53469\,\eta_1 - 3.21395 = 0$$

$$F'(\eta_1) = 3\eta_1^2 - 8.85634\,\eta_1 + 6.53469$$

$F(1.55) = +0.000017$; $F'(1.55) = 0.01486$

$F(1.54886) =$ $\boxed{\eta_1 = 1.54886}$

$$\eta_1^2 - 2.87931\,\eta_1 + 2.07504 = 0$$

$$\eta_1 = 1.43966 \pm \sqrt{-0.00242}$$

$\sqrt{\frac{\rho}{\rho_E}} = 1.6$ $\frac{\rho}{\rho_E} = 2.56$

$\eta_1 = -0.124576$.	$\eta_2 = +1.47913$
$\eta_1 = -0.051777$.	$\eta_2 = +0.51718$
$\eta_1 = +0.465896$	$\eta_2 = -0.10392$
$\eta_1 = +0.606561$	$\eta_2 = +0.80608$
$\eta_1 = +0.675718$.	$\eta_2 = +1.29695$
$\eta_1' = +1.54886$.	$\eta_2 = -0.64007$

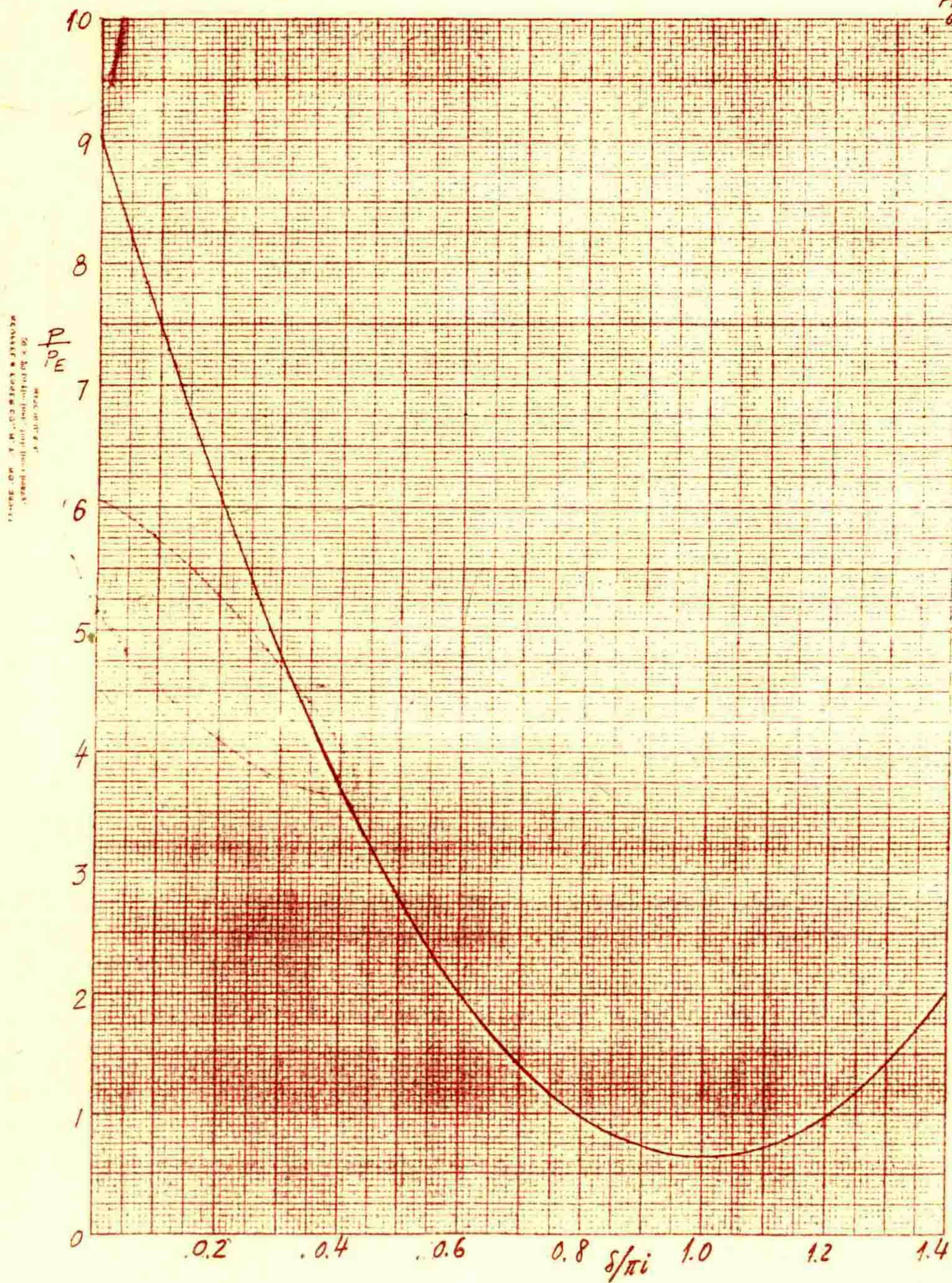


Fig. 4
10
9
8
7
6
5
4
3
2
1
0
P/P_E
.0.2
.0.4
.0.6
0.8
$\delta/\pi i$
1.0
1.2
1.4

$$\xi_1 - \xi_2 = (A-B)\left[27.000\,(\xi_1 - \xi_2) - 5640\,(\xi_1^2 - \xi_2^2) + 282000\,(\xi_1^3 - \xi_2^3)\right] \qquad \xi_1 - \xi_2 = \zeta$$

$$\xi_1 + \xi_2 = (A+B)\left[27.000\,(\xi_1 + \xi_2) - 5640\,(\xi_1^2 + \xi_2^2) + 282000\,(\xi_1^3 + \xi_2^3)\right] \qquad \xi_1 + \xi_2 = \eta$$

$$\frac{\zeta + \eta}{2} = \xi_1 \qquad \frac{\eta - \zeta}{2} = \xi_2$$

$$1 = (A-B)\left[27.000 - 5640\,\eta + \frac{282000}{4}\left\{(\eta+\zeta)^2 + (\eta^2 - \zeta^2) + (\eta - \zeta)^2\right\}\right]$$

$$1 = (A-B)\left[27.000 - 5640\,\eta + 70500\,(3\eta^2 + \zeta^2)\right]$$

$$\eta = (A+B)\left[27.000\,\eta - 2820\,(\eta^2 + \zeta^2) + 70500\,(\eta^3 + 3\eta\zeta^2)\right]$$

$$\boxed{F(\xi) = \xi\left[\alpha - 2\beta\xi + 4\gamma\xi^2\right]}$$

$$\boxed{\begin{aligned} 1 &= (A-B)\left[\alpha - 2\beta\eta + \gamma\,(3\eta^2 + \zeta^2)\right] \\ \eta &= (A+B)\left[\alpha\eta - \beta\,(\eta^2 + \zeta^2) + \gamma\,(\eta^3 + 3\eta\zeta^2)\right] \end{aligned}}$$

$\alpha = 27.00$

$\beta = 2820$

$\gamma = 70500$

$$\gamma \zeta^2 = \frac{1}{A-B} - \alpha + 2\beta\eta - 3\gamma\eta^2$$

$$\gamma\eta = (A+B)\Big[(\alpha\gamma)\eta - (\beta\gamma)\eta^2 - \frac{\beta}{A-B} + \alpha\beta - 2\beta^2\eta + 3\beta\gamma\eta^2 + \gamma^2\eta^3 + \left(\frac{3\gamma}{A-B}\right)\eta - (3\alpha\gamma)\eta + (6\beta\gamma)\eta^2 - 9\gamma^2\eta^3\Big]$$

$$(-8\gamma^2)\eta^3 + (8\beta\gamma)\eta^2 + \left\{-2\alpha\gamma - 2\beta^2 + \frac{3\gamma}{A-B} - \frac{\gamma}{A+B}\right\}\eta + \left\{\alpha\beta - \frac{\beta}{A-B}\right\} = 0$$

or
$$\eta^3 - \left(\frac{\beta}{\gamma}\right)\eta^2 + \left\{\frac{(\alpha\gamma+\beta^2)}{4\gamma^2} - \frac{1}{8\gamma}\left(\frac{3}{A-B} - \frac{1}{A+B}\right)\right\}\eta - \frac{\beta}{8\gamma^2}\left(\alpha - \frac{1}{A-B}\right) = 0$$

$$\boxed{\eta^{*3} - 4\eta^{*2} + \left\{4.95745 - \frac{1}{56.4}\left(\frac{3}{A-B} - \frac{1}{A+B}\right)\right\}\eta^* - \left\{1.91489 - \frac{1}{14.1}\frac{1}{A-B}\right\} = 0}$$

$$\boxed{\zeta^{*2} = \left\{\frac{1}{70.5(A-B)} - 3.829787\right\} + 8\eta^* - 3\eta^{*2}}$$

$$\eta^* = 100\eta = \frac{\delta_1+\delta_2}{\pi i} - \frac{1}{\pi i}\frac{\delta_1+\delta_2}{l} = 100\frac{\delta_1+\delta_2}{l}$$

$$\zeta^* = 100\zeta$$

p/p_E	$A-B$	$\frac{1}{A-B}$	α	β	γ		
9.0	0.012345	81.005	1.12740	-3.83015	7.6603		
8.41	0.014740	67.843	1.79103	-2.89667	5.7933		
7.84	0.017325	57.720	2.29449	-2.17873	4.3574		
7.29	0.020782	48.119	2.77163	-1.49781	2.9956		
6.76	0.025344	39.457	3.20022	-0.88348	1.7669		
6.25	0.031691	31.555	3.58972	-0.32305	0.64610		
5.76	0.041190	24.278	3.94733	+0.19305	-0.38610		
5.29	0.057001	17.544	4.27736	+0.67064	-1.34128		
4.84	0.088634	11.282	4.58358	1.11475	-2.2295		
4.41	0.183575	5.4474	4.86833	1.52855	-3.0571		
4.00	∞	0	5.13376	1.91489	-3.8298		
3.61	-0.196508	-5.0889	5.38136	2.27580	-4.5516		
3.24	-0.101606	-9.8419	5.61233	2.61290	-5.2258		
2.89	-0.070037	-14.2782	5.82768	2.92753	-5.8551		
2.56	-0.054311	-18.4125	6.02817	3.22074	-6.4415		
2.25	-0.044932	-22.2559	6.21440	3.49332	-6.9867		
1.96	-0.038730	-25.820	6.38698	3.74610	-7.4922		
1.69	-0.034351	-29.111	6.54622	3.97950	-7.9590		
1.44	-0.031115	-32.139	6.69266	4.19425	-8.3885		
1.21	-0.028641	-34.915	6.82689	4.39113	-8.7823		
1.00	-0.026721	-37.424	6.94809	4.56907	-9.1382		
0.81	-0.025191	-39.697	7.05791	4.73028	-9.4606		
0.64	-0.023988	-41.688	7.15391	4.87149	-9.7430		
0.49							

$\frac{P}{P_E} = \underline{9.00}$;

$$\eta^{*3} - 4\eta^{*2} + 1.12740\eta^* + 3.83015 = 0$$

$$F'(\eta^*) = 3\eta^{*2} - 8\eta^* + 1.12740$$

$F(1.477) = -0.00868$; $F'(1.477) = -4.144$

$F(1.47491) =$ 0.K.

$$\eta^{*2} - 2.52509\eta^* - 2.59188 = 0$$

$\eta^* = 1.47491$; $\zeta^{*2} = 12.93350$; $\zeta^* = 3.59632$; $\begin{cases} \xi_1^* = 2.53562 \\ \xi_2^* = -1.06070 \end{cases}$

$$\eta^* = 1.26255 \pm \sqrt{4.19091} = \begin{matrix} +3.30972 \\ -0.78463 \end{matrix}$$

$\eta^* = +3.30972$; $\zeta^{*2} = 1.27532$; $\zeta^* = 1.12930$; $\begin{cases} \xi_1^* = 2.21951 \\ \xi_2^* = 1.09021 \end{cases}$

$\frac{P}{P_E} = 7.84$ $\quad \eta^{*3} - 4\eta^{*2} + 2.29449\eta^* + 2.17873 = 0$

$$F'(\eta^*) = 3\eta^{*2} - 8\eta^* + 2.29449$$

$F(1.50) = -0.00454 \; ; \qquad F'(1.50) = -2.95551$

$\eta^* = 1.49846 \; , \quad \zeta^{*2} = 9.60893 \; ; \quad \zeta^* = 3.09983 \; ; \quad \begin{cases} \zeta_1^* = 2.29915 \\ \zeta_2^* = -0.80068 \end{cases}$

$\eta^{*2} - 2.50154\,\eta^* - 1.45397 = 0$

$\eta^* = 1.25077 \pm \sqrt{3.01870} = \begin{matrix} 2.98813 \\ -0.48659 \end{matrix}$

$\eta^* = 2.98813 \; ; \quad \zeta^{*2} = 1.47568 , \quad \zeta^* = 1.21477 \; ; \quad \begin{cases} \zeta_1^* = 2.10145 \\ \zeta_2^* = 0.88668 \end{cases}$

$\frac{P}{P_E} = 6.76$ $\quad \eta^{*3} - 4\eta^{*2} + 3.20022\eta^* + 0.88348 = 0$

$F'(\eta^*) = 3\eta^{*2} - 8\eta^* + 3.20022$

$F(1.53) = -0.00221 \; ; \qquad F'(1.53) = 2.01708$

$\eta^* = 1.52890 \; ; \quad \zeta^{*2} = 6.98549 \; ; \quad \zeta^* = 2.64301 \; ; \quad \begin{cases} \zeta_1^* = 2.08596 \\ \zeta_2^* = -0.55705 \end{cases}$

$\eta^{*2} - 2.47110\,\eta^* - 0.57784 = 0$

$\eta^* = 1.23555 \pm \sqrt{2.10442} = \begin{matrix} +2.68621 \\ -0.21511 \end{matrix}$

$\eta^* = 2.68621 \; ; \quad \zeta^{*2} = 1.60941 \; ; \quad \zeta^* = 1.26862 \; : \quad \begin{cases} \zeta_1^* = 1.97742 \\ \zeta_2^* = 0.70880 \end{cases}$

$\frac{P}{P_E} = 5.76 \quad \eta^{*3} - 4\eta^{*2} + 3.94733\,\eta^* - 0.19305 = 0$

$F'(\eta^*) = 3\eta^{*2} - 8\eta^* + 3.94733$

$F(1.58) = +0.00244\,; \quad F'(1.58) = 1.20347$

$\eta^* = 1.58203\,; \quad \zeta^{*2} = 4.76168\,; \quad \zeta^* = 2.18213\,; \quad \begin{cases} \xi_1^* = 1.88208 \\ \xi_2^* = -0.30005 \end{cases}$

$\eta^{*2} - 2.41797\,\eta^* + 0.12203 = 0$

$\eta^* = 1.20899 \pm \sqrt{1.33963} = \begin{matrix} 2.36641 \\ 0.05156 \end{matrix}$

$\eta^* = 2.36641\,; \quad \zeta^{*2} = 1.74549\,; \quad \zeta^* = 1.32117\,; \quad \begin{cases} \xi_1^* = 1.84379 \\ \xi_2^* = 0.52262 \end{cases}$

$\frac{P}{P_E} = 4.84 \quad \eta^{*3} - 4\eta^{*2} + 4.58358\,\eta^* - 1.11475 = 0$

$F'(\eta^*) = 3\eta^{*2} - 8\eta^* + 4.58358$

$F(0.33) = -0.00183\,; \quad F'(0.33) = 2.27028$

$\eta^* = 0.33081$

$\eta^{*2} - 3.66919\,\eta^* + 3.36978 = 0$ } No useful root !!!

$\frac{P}{P_E} = 5.29 \quad \eta^{*3} - 4\eta^{*2} + 4.27736\,\eta^* - 0.67064 = 0$

$F'(\eta^*) = 3\eta^{*2} - 8\eta^* + 4.27736\,; \quad F(0.19) = +0.00452\,; F'(0.19) = 2.8657$

$\eta^* = 0.18842$

$\eta^{*2} - 3.81158\,\eta^* + 3.55918 = 0\,; \quad \eta^* = 1.90579 \pm \sqrt{0.0727555} = \begin{matrix} 2.17571 \\ 1.63587 \end{matrix}$

$\eta^* = 2.17571\,; \quad \zeta^{*2} = 1.86326\,; \quad \zeta^* = 1.36501\,; \quad \begin{matrix} \xi_1^* = 1.77036 \\ \xi_2^* = 0.40535 \end{matrix}$

$\eta^* = 1.63587\,; \quad \zeta^{*2} = 3.71747\,; \quad \zeta^* = 1.92808\,; \quad \xi_1^* = 1.78198\,; \quad \xi_2^* = -0.14610$

$\frac{P}{P_E} = 1.44 \qquad \eta^{*3} - 1.6000\,\eta^{*2} - 0.32103\,\eta^{*} + 3.35541 = 0$

$F(\eta^*) = 3\eta^{*2} - 3.2000\,\eta^{*} - 0.32103$

Impossible!

$$C = \frac{3}{4\pi^2}\left\{ \sum_{n=1,2,3}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E} - n^2\right]^2} - \frac{1}{729}\sum_{n=1,2,3}^{\infty} \frac{1}{n^2\left[\frac{P}{9P_E} - n^2\right]^2} \right\}$$

Consider

$$\sum_{n=1,2,3}^{\infty} \frac{1}{n^2\left[\frac{P}{P_E} - n^2\right]^2} = \frac{1}{\left(\frac{P}{P_E}\right)^2} \sum_{n=1,2,3}^{\infty}\left\{ \frac{1}{n^2} + \frac{1}{\frac{P}{P_E} - n^2} \right\}$$

$$- \frac{1}{\left(\frac{P}{P_E}\right)} \frac{\partial}{\partial\left(\frac{P}{P_E}\right)} \sum_{n=1,2,3}^{\infty} \frac{1}{\left[\frac{P}{P_E} - n^2\right]}\Bigg\}$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left\{ \frac{\pi^2}{6} + \frac{\pi}{2\sqrt{\frac{P}{P_E}}}\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{2\left(\frac{P}{P_E}\right)} \right\}$$

$$- \frac{1}{\left(\frac{P}{P_E}\right)} \frac{\partial}{\partial\left(\frac{P}{P_E}\right)}\left\{ \frac{\pi}{2\sqrt{\frac{P}{P_E}}}\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{2\frac{P}{P_E}} \right\}$$

$$\frac{\partial\left(\frac{P}{P_E}\right)^{\frac{1}{2}}}{\partial\left(\frac{P}{P_E}\right)} = \frac{1}{2}\frac{1}{\left(\frac{P}{P_E}\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left\{ \frac{\pi^2}{6} + \frac{\pi}{2\sqrt{\frac{P}{P_E}}}\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{2\left(\frac{P}{P_E}\right)} - \frac{1}{2}\left(\frac{P}{P_E}\right)^{\frac{1}{2}}\left[-\frac{\pi}{2\left(\frac{P}{P_E}\right)}\cot\pi\sqrt{\frac{P}{P_E}} \right.\right.$$

$$\left.\left. - \frac{\pi^2}{2\sqrt{\frac{P}{P_E}}}\csc^2\pi\sqrt{\frac{P}{P_E}} + \frac{1}{\left(\frac{P}{P_E}\right)^{3/2}} \right]\right\}$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left\{ \frac{\pi^2}{6} + \frac{3\pi}{4\sqrt{\frac{P}{P_E}}}\cot\pi\sqrt{\frac{P}{P_E}} + \frac{1}{4}\pi^2\csc^2\pi\sqrt{\frac{P}{P_E}} - \frac{1}{\left(\frac{P}{P_E}\right)} \right\}$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)^2}\left\{ \frac{5}{12}\pi^2 + \frac{\pi^2}{4}\cot^2\pi\sqrt{\frac{P}{P_E}} + \frac{3\pi}{4\sqrt{\frac{P}{P_E}}}\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{\frac{P}{P_E}} \right\}$$

65

$$C=\frac{3}{4\pi^2\left(\frac{P}{P_E}\right)^2}\left\{\frac{\pi^2}{4}\left(\cot^2\pi\sqrt{\frac{P}{P_E}}-\frac{1}{9}\cot^2\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right)+\frac{3\pi}{4\sqrt{\frac{P}{P_E}}}\left(3\cot\pi\sqrt{\frac{P}{P_E}}-\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right)\right.$$
$$\left.+\frac{10}{27}\pi^2\right\}$$

$$C=\frac{1}{\left(\frac{P}{P_E}\right)^2}\left\{\frac{3}{16}\left(\cot^2\pi\sqrt{\frac{P}{P_E}}-\frac{1}{9}\cot^2\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right)+\frac{9}{16\pi\sqrt{\frac{P}{P_E}}}\left(\cot\pi\sqrt{\frac{P}{P_E}}-\frac{1}{3}\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right)\right.$$
$$\left.+\frac{5}{18}\right\}$$

$$D=-\frac{2}{\pi^2}\sum_{n=1,2,3}^{\infty}(-1)^n\frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E}-n^2\right]^2}=-\frac{2}{\pi^2}\left\{\sum_{n=1,2,3}^{\infty}\frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E}-n^2\right]^2}-2\sum_{m=1,2,3}^{\infty}\frac{\sin^2\frac{2m\pi}{3}}{(2m)^2\left[\frac{P}{P_E}-(2m)^2\right]^2}\right.$$

$$=\frac{2}{\pi^2}\left\{\sum_{n=1,2,3}^{\infty}\frac{\sin^2\frac{n\pi}{3}}{n^2\left[\frac{P}{P_E}-n^2\right]^2}-\frac{2}{64}\sum_{m=1,2,3}^{\infty}\frac{\sin^2\frac{m\pi}{3}}{m^2\left[\frac{P}{4P_E}-m^2\right]^2}\right\}$$

$$=\frac{6}{4\pi^2\left(\frac{P}{P_E}\right)^2}\left\{\frac{\pi^2}{4}\left(\cot^2\pi\sqrt{\frac{P}{P_E}}-\frac{1}{2}\cot^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)-\frac{\pi^2}{36}\left(\cot^2\frac{\pi}{3}\sqrt{\frac{P}{P_E}}-\frac{1}{2}\cot^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)\right.$$
$$+\frac{\pi}{4\sqrt{\frac{P}{P_E}}}\left(3\cot\pi\sqrt{\frac{P}{P_E}}-3\cot\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)-\frac{\pi}{4\sqrt{\frac{P}{P_E}}}\left(\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}-\cot\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)$$
$$\left.+\frac{5}{27}\pi^2\right\}$$

$$D = \frac{1}{\left(\frac{P}{P_E}\right)^2}\Bigg\{ \frac{3}{8}\left(\cot^2 \pi\sqrt{\frac{P}{P_E}} - \frac{1}{2}\cot^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right) - \frac{1}{24}\left(\cot^2\frac{\pi}{3}\sqrt{\frac{P}{P_E}} - \frac{1}{2}\cot^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)$$

$$+ \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\left(3\cot\pi\sqrt{\frac{P}{P_E}} - 3\cot\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right) - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\left(\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}} - \cot\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)$$

$$+ \frac{5}{18}\Bigg\}$$

But $\cot 2\theta - \cot\theta = \dfrac{\cos^2\theta - \sin^2\theta - 2\cos^2\theta}{2\sin\theta\cos\theta} = -\dfrac{1}{\sin 2\theta}$

$$\cot^2 2\theta - \frac{1}{2}\cot^2\theta = \frac{(\cos^2\theta - \sin^2\theta)^2 - 2\cos^4\theta}{4\sin^2\theta\cos^2\theta}$$

$$= \frac{\sin^4\theta - 2\cos^2\theta\sin^2\theta - \cos^4\theta}{4\sin^2\theta\cos^2\theta}$$

$$= \frac{\sin^4\theta - 2\cos^2\theta\sin^2\theta - (1 - 2\sin^2\theta + \sin^4\theta)}{4\sin^2\theta\cos^2\theta}$$

$$= -\frac{1}{2} - \frac{1 - 2\sin^2\theta}{4\sin^2\theta\cos^2\theta} = -\frac{1}{2} - \cot 2\theta\cdot\frac{1}{\sin 2\theta}$$

$$D = \frac{1}{\left(\frac{P}{P_E}\right)^2}\left\{ -\frac{3}{8}\left(\frac{1}{2} + \frac{\cot \pi\sqrt{\frac{P}{P_E}}}{\sin \pi\sqrt{\frac{P}{P_E}}}\right) + \frac{1}{24}\left(\frac{1}{2} + \frac{\cot \frac{\pi}{3}\sqrt{\frac{P}{P_E}}}{\sin \frac{\pi}{3}\sqrt{\frac{P}{P_E}}}\right)\right.$$

$$\left. - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\left(\frac{3}{\sin \pi\sqrt{\frac{P}{P_E}}}\right) + \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\left(\frac{1}{\sin \frac{\pi}{3}\sqrt{\frac{P}{P_E}}}\right) + \frac{5}{18}\right\}$$

$$\boxed{D = \frac{1}{\left(\frac{P}{P_E}\right)^2}\left\{ \frac{1}{9} - \frac{3}{8}\left(\frac{\cot \pi\sqrt{\frac{P}{P_E}}}{\sin \pi\sqrt{\frac{P}{P_E}}} - \frac{1}{9}\,\frac{\cot \frac{\pi}{3}\sqrt{\frac{P}{P_E}}}{\sin \frac{\pi}{3}\sqrt{\frac{P}{P_E}}}\right) - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\left(\frac{3}{\sin \pi\sqrt{\frac{P}{P_E}}} - \frac{1}{\sin \frac{\pi}{3}\sqrt{\frac{P}{P_E}}}\right)\right\}}$$

$$\left.\begin{aligned} \xi_1 &= A\left(\frac{F_1}{P_E}\right) + B\left(\frac{F_2}{P_E}\right) \\ \xi_2 &= B\left(\frac{F_1}{P_E}\right) + A\left(\frac{F_2}{P_E}\right) \end{aligned}\right\}$$

$$\xi_1 A - B\xi_2 = (A^2 - B^2)\frac{F_1}{P_E}$$

$$\therefore \quad \frac{F_1}{P_E} = \frac{A\xi_1 - B\xi_2}{A^2 - B^2}$$

$$B\xi_1 - A\xi_2 = (B^2 - A^2)\frac{F_2}{P_E}$$

$$\therefore \quad \frac{F_2}{P_E} = \frac{-B\xi_1 + A\xi_2}{A^2 - B^2}$$

Or

$$\boxed{\begin{aligned} \left(\frac{F_1}{P_E}\right) &= \xi_1^*\left[0.27 - 0.5640\,\xi_1^* + 0.28200\,\xi_1^{*2}\right] \\ \left(\frac{F_2}{P_E}\right) &= \xi_2^*\left[0.27 - 0.5640\,\xi_2^* + 0.28200\,\xi_2^{*2}\right] \end{aligned}}$$

68

$$(A-B)=\frac{1}{\left(\frac{P}{P_E}\right)}\left[\frac{1}{9}+\frac{1}{4\pi\sqrt{\frac{P}{P_E}}}\left\{3\left(\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}+2\cot\pi\sqrt{\frac{P}{P_E}}\right)-\left(\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}+2\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right)\right\}\right]$$

$$\tan\theta+2\cot 2\theta=\frac{2\sin^2\theta+2(\cos^2\theta-\sin^2\theta)}{2\sin\theta\cos\theta}=\cot\theta$$

$$\boxed{(A-B)=\frac{1}{\left(\frac{P}{P_E}\right)}\left[\frac{1}{9}+\frac{1}{4\pi\sqrt{\frac{P}{P_E}}}\left(3\cot\frac{\pi}{2}\sqrt{\frac{P}{P_E}}-\cot\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)\right]}$$

Near $\frac{P}{P_E}=4$; $\quad \sqrt{\frac{P}{P_E}}=2+\varepsilon$

$\frac{1}{2}-\frac{1}{6}$

$\frac{3-1}{6}$

$$\underline{H=0.100563}=A+B$$

$$(A-B)=\frac{1}{4}\left[\frac{1}{4\pi\cdot 2}\,3\cot\left(\pi+\frac{\pi\varepsilon}{2}\right)\right]$$

$$=\frac{3}{32\pi}\cot\frac{\pi\varepsilon}{2}=\frac{3}{32\pi}\,\frac{1-\frac{\pi^2\varepsilon^2}{2!\,4}}{\frac{\pi\varepsilon}{2}\left(1-\frac{\pi^2\varepsilon^2}{3!\,4}+\cdots\right)}$$

$$=\frac{3}{16\pi^2}\,\frac{1}{\varepsilon}\left(1-\frac{1}{3}\frac{\pi^2\varepsilon^2}{4}+\cdots\right)$$

$$(A^2-B^2)^2=(A+B)^2(A-B)^2=\left(\frac{0.301689}{16\pi^2}\right)^2\frac{1}{\varepsilon^2}\left(1-\frac{1}{3}\frac{\pi^2\varepsilon^2}{2}+\cdots\right)$$

$$\frac{\varepsilon}{l} = C\left\{\left(\frac{F_1}{P_E}\right)^2 + \left(\frac{F_2}{P_E}\right)^2\right\} + D\,\frac{F_1 F_2}{P_E^2}$$

$$= \frac{C}{(A^2-B^2)^2}\left\{(A^2+B^2)(\xi_1^2+\xi_2^2) - 4AB\,\xi_1\xi_2\right\} + \frac{D}{(A^2-B^2)^2}\left\{(A^2+B^2)\xi_1\xi_2 - AB(\xi_1^2+\xi_2^2)\right\}$$

$$\boxed{\sqrt{\frac{P}{P_E}} = 2+\varepsilon\,;}$$

$$C = \frac{3}{256}\cot^2\pi\varepsilon = \frac{3}{256}\,\frac{\left(1-\frac{1}{3}\pi^2\varepsilon^2+\right)^2}{\pi^2\varepsilon^2}$$

$$= \frac{3}{256\pi^2}\,\frac{1}{\varepsilon^2}\left(1-\frac{2}{3}\pi^2\varepsilon^2+\cdots\right)$$

$$\frac{4}{6}-\frac{1}{6}$$

$$\therefore\quad \frac{C}{(A^2-B^2)^2} = \frac{3\ \pi^2}{(0.301689)^2-}\;\frac{1-\frac{2}{3}\pi^2\varepsilon^2+--}{1-\frac{1}{6}\pi^2\varepsilon^2+--}$$

$$= \frac{3\pi^2}{(0.301689)^2}\left(1-\frac{1}{2}\pi^2\varepsilon^2+--\right)$$

$$D = -\frac{3}{8\cdot 16}\;\frac{1-\frac{1}{3}\pi^2\varepsilon^2+\cdots}{\pi^2\varepsilon^2\left(1-\frac{1}{6}\pi^2\varepsilon^2+--\right)} = +\frac{6}{16^2\cdot\pi^2}\,\frac{1}{\varepsilon^2}\left(1-\frac{1}{6}\pi^2\varepsilon^2+\cdots\right)$$

$$\frac{D}{(A^2-B^2)^2} = -\frac{6\pi^2}{(0.301689)^2}\left(1-\frac{1}{6}\pi^2\varepsilon^2+---\right)$$

$$\frac{\varepsilon}{l} = \frac{3\pi^2}{(0.301689)^2}\left[(A^2+B^2)(\xi_1-\xi_2)^2 + 2AB(\xi_1-\xi_2)^2\right] = \frac{3\pi^2(\xi_1-\xi_2)^2}{(0.301689)^2}(A+B)^2$$

Let $\sqrt{\frac{P}{P_E}} = \underline{3-\varepsilon}$.

$$C = \frac{1}{81}\left\{\frac{3}{16}\left[\cot^2\pi(3-\varepsilon) - \frac{1}{9}\cot^2\frac{\pi}{3}(3-\varepsilon)\right] + \frac{3}{16\pi}\left[\cot\pi(3-\varepsilon) - \frac{1}{3}\cot\frac{\pi}{3}(3-\varepsilon)\right] + \frac{5}{18}\right\}$$

Now
$$\cot\pi(3-\varepsilon) = -\cot\pi\varepsilon = -\frac{(1-\frac{1}{2!}\pi^2\varepsilon^2)\ldots}{\pi\varepsilon(1-\frac{1}{3!}\pi^2\varepsilon^2+\ldots)}$$
$$= -\frac{1}{\pi\varepsilon}\left(1-\frac{1}{3}\pi^2\varepsilon^2+\ldots\right)$$

$$\cot\frac{\pi}{3}(3-\varepsilon) = -\cot\frac{\pi\varepsilon}{3} = -\frac{3}{\pi\varepsilon}\left(1-\frac{1}{3}\frac{\pi^2\varepsilon^2}{9}+\ldots\right)$$

$$\therefore \quad \cot\pi(3-\varepsilon) - \frac{1}{3}\cot\frac{\pi}{3}(3-\varepsilon) = +\frac{1}{\pi\varepsilon}\frac{1}{3}\left[\frac{8}{9}\pi^2\varepsilon^2\right] \to 0 \text{ as } \varepsilon\to 0.$$

But
$$\cot^2\pi(3-\varepsilon) - \frac{1}{9}\cot^2\frac{\pi}{3}(3-\varepsilon) = \frac{1}{\pi^2\varepsilon^2}\left[\left(1-\frac{2}{3}\pi^2\varepsilon^2+\ldots\right) - \left(1-\frac{2}{3}\frac{\pi^2\varepsilon^2}{9}+\ldots\right)\right]$$
$$= -\frac{2}{3}\cdot\left(\frac{8}{9}\right) = -\frac{16}{27}$$

$$\therefore \quad C = \frac{1}{81}\cdot\left\{-\frac{1}{9}+\frac{5}{18}\right\} = \frac{1}{81}\cdot\frac{3}{18} = \underline{\frac{1}{81}\cdot\frac{1}{6}}$$

$$\frac{\cot\pi\sqrt{\frac{P}{P_E}}}{\sin\pi\sqrt{\frac{P}{P_E}}} = \frac{\cot\pi(3-\varepsilon)}{\sin\pi(3-\varepsilon)} = -\frac{1-\frac{1}{3}\pi^2\varepsilon^2+\ldots}{\pi^2\varepsilon^2(1-\frac{1}{3!}\pi^2\varepsilon^2+\ldots)} = -\frac{1-\frac{1}{6}\pi^2\varepsilon^2+\ldots}{\pi^2\varepsilon^2}$$

$$\frac{\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}}{\sin\frac{\pi}{3}\sqrt{\frac{P}{P_E}}} = -\frac{1-\frac{1}{6}\frac{\pi^2\varepsilon^2}{9}}{\frac{\pi^2\varepsilon^2}{9}} \quad \Bigg| \quad \therefore D = \frac{1}{81}\left\{\frac{1}{9}-\frac{3}{8}\cdot\frac{1}{6}\cdot\frac{8}{9}\right\} = \underline{\frac{1}{81}\cdot\frac{1}{18}}$$

$\frac{3}{16} = 0.187500$; $\frac{9}{16\pi} = 0.179049$; $\frac{5}{18} = 0.2777778$

①	②	③	④	⑤	⑥	⑦	⑧	⑨
P/P_E	$\sqrt{\frac{P}{P_E}}$	$\cot \pi \sqrt{\frac{P}{P_E}}$	$\sin \pi \sqrt{\frac{P}{P_E}}$	$\cot \frac{\pi}{3} \sqrt{\frac{P}{P_E}}$	$\sin \frac{\pi}{3} \sqrt{\frac{P}{P_E}}$	③ + $\frac{1}{3}$ ⑤	③ − $\frac{1}{3}$ ⑤	$\left(\frac{P}{P_E}\right)^2$
9.00	3.00							81.0000
7.84	2.80	−1.3764	0.58779	−4.7046	0.20791	−2.94460	+0.19180	61.4656
6.76	2.60	−0.32492	0.95106	−2.2460	0.40674	−1.07179	+0.42195	45.6976
5.76	2.40	0.32492	0.95106	−1.3764	0.58779	−0.13388	+0.78372	33.1776
5.29	2.30	0.72654	0.80902	−1.1106	0.66913	+0.35634	+1.09674	27.9841

$\frac{3}{8} = 0.375$, $\frac{3}{8\pi} = 0.119366$

⑩	⑪	⑫	⑬	⑭	⑮	⑯	⑰	⑱
C	$9 \sin \frac{\pi}{3} \sqrt{\frac{P}{P_E}}$	③/④ − ⑤/⑪	3/④ − 1/⑥	D	$\left(\frac{F_1}{P_E}\right)_1$	$\left(\frac{F_2}{P_E}\right)_1$	$\left(\frac{F_1}{P_E}\right)_2$	$\left(\frac{F_2}{P_E}\right)_2$
0.0020576				+0.0006859	0.90421	−0.01058	1.65571	−1.25247
0.0049959	1.87119	0.17258	0.29409	+0.0005508	0.69373	−0.00743	1.06671	−0.72251
0.0045589	3.66066	0.27191	0.69580	−0.0004989	0.50899	+0.00844	0.66868	−0.37416
0.0095418	5.29011	0.60182	1.45309	−0.0056316	0.34807	+0.02732	0.39038	−0.13941
0.0155958	6.02247	1.08247	2.21371	−0.0146406	0.27503	+0.03556	0.28590	−0.05237

$$\frac{l}{\pi_i}\frac{F_1}{P_E} = \xi_1^*\left[27.00 - 56.40\,\xi_1^* + 28.20\,\xi_1^{*2}\right]$$

O.K.

17

[illegible] incorrect !!!

⑲	⑳	㉑	㉒	㉓	㉔	㉕	㉖	㉗	㉘
⑮^2 + ⑯^2	⑮⑯	⑰^2 + ⑱^2	⑰⑱	$(\varepsilon/l)_{400}$	$\lvert\varepsilon/l\rvert_{400}$	$(\varepsilon_{TOT})_1$	$(\varepsilon_{TOT})_2$	$\left[\frac{\varepsilon/l_{TOT}}{(\pi\frac{i}{l})^2}\right]_1$	$\left[\frac{\varepsilon/l_{TOT}}{\pi^2(\frac{i}{l})^2}\right]_2$
0.81771	-0.009567	4.32261	-2.08201	0.67038	2.98646	9.670[illegible]8	11.9[illegible]64[illegible]	25.7595	83.6615
0.48132	-0.005154	1.65919	-0.77071	0.57566	1.81934	8.41566	8.65934	22.2315	53.3235
0.25914	+0.004296	0.58713	-0.25019	0.50280	1.19105	7.26[illegible]20	7.75105	19.3300	36.5363
0.12190	+0.009509	0.17183	-0.054423	0.44384	0.77842	6.20354	6.53842	16.8560	25.2205
0.07671	+0.009760	0.08448	-0.014973	0.42252	0.61468	5.71252	5.90[illegible]18	15.1530	20.6570

Corrected for

$$\frac{\pi i}{l} = \frac{1}{100}$$

Symmetrical Case!

(1) Bending Energy,

$$\frac{EI}{4} l \sum_{n=1,3,5}^{\infty} \left(\frac{n\pi}{l}\right)^4 a_n^2 = W_1$$

(2) Spring Energy, $\int_0^{\delta} F\,d\delta$

$$P_E \int_0^{\delta} \left(\frac{F}{P_E}\right) d\delta = P_E \cdot l \int_0^{\xi} \left(\frac{F}{P_E}\right) d\xi$$

$$= P_E l \left[\frac{27.000}{2} \xi^2 - \frac{5640}{3} \xi^3 + \frac{282000}{4} \xi^4 \right] = W_2$$

(3) Compression Energy

$$\frac{P^2 l}{2EA} = \frac{P_E^2 l}{2EA} \left(\frac{P}{P_E}\right)^2 = \frac{1}{2}\left(\frac{\pi^2 EI}{l^2}\right)^2 \frac{l}{EA} \left(\frac{P}{P_E}\right)^2$$

$$= \frac{1}{2} \frac{\pi^4 EI^2}{l^3 A} \left(\frac{P}{P_E}\right)^2 = \frac{\pi^4 EI}{2 l^3} k^2 \left(\frac{P}{P_E}\right)^2$$

$$= \frac{l}{2} \cdot P_E \pi^2 \left(\frac{k}{l}\right)^2 \left(\frac{P}{P_E}\right)^2 = W_3$$

$$W_1 = \frac{\pi^4 EI}{4 l^2} l \sum n^4 \left(\frac{a_n}{l}\right)^2 = \frac{\pi^2 P_E}{2} \frac{l}{2} \sum n^4 \left(\frac{a_n}{l}\right)^2$$

$$\frac{W_1}{l P_E} = \frac{4}{\pi^2} \left(\frac{F_1}{P_E}\right)^2 \sum_{n=1,3,5}^{\infty} \frac{\sin^2 \frac{n\pi}{3}}{\left[\frac{P}{P_E} - n^2\right]^2} ; \quad \frac{W_3}{P_E l} = \frac{\pi^2 \left(\frac{k}{l}\right)^2}{2} \left(\frac{P}{P_E}\right)^2$$

$$\frac{W_2}{P_E l} = 2(13.500 \xi^2 - 1880 \xi^3 + 70500 \xi^4)$$

$$\frac{W_1}{P_E l} = \frac{3}{\pi^2}\left(\frac{F_1}{P_E}\right)^2 \left[\sum_{n=1,3,5}^{\infty} \frac{1}{\left(\frac{P}{P_E} - n^2\right)^2} - 81 \sum_{n=1,3,5} \frac{1}{\left[\frac{P}{9P_E} - n^2\right]^2} \right]$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{\left(\frac{P}{P_E} - n^2\right)^2} = -\frac{\partial}{\partial\left(\frac{P}{P_E}\right)} \sum_{n=1,3,5}^{\infty} \frac{1}{\frac{P}{P_E} - n^2}$$

$$= +\frac{1}{2}\frac{1}{\sqrt{\frac{P}{P_E}}} \frac{\partial}{\partial\sqrt{\frac{P}{P_E}}} \frac{\pi}{4\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$= \frac{\pi}{8\sqrt{\frac{P}{P_E}}} \frac{\partial}{\partial\sqrt{\frac{P}{P_E}}} \frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\sqrt{\frac{P}{P_E}}} = \frac{\pi}{8\sqrt{\frac{P}{P_E}}} \left\{ -\frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\left(\frac{P}{P_E}\right)} + \frac{\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\sqrt{\frac{P}{P_E}}}\frac{\pi}{2} \right\}$$

$$= \frac{\pi^2}{16\left(\frac{P}{P_E}\right)} \left\{ \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{2}{\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \right\}$$

$$\boxed{\frac{W_1}{P_E l} = \frac{3}{16\left(\frac{P}{P_E}\right)}\left(\frac{F_1}{P_E}\right)^2 \left\{ \left(\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\sec^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right) - \frac{2}{\pi\sqrt{\frac{P}{P_E}}}\left(\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\tan\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right) \right\}}$$

When $\sqrt{\frac{P}{P_E}} = 3 - \varepsilon$

$$\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\sec^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}} = \frac{1}{\cos^2\left(\frac{3}{2}\pi - \frac{\varepsilon}{2}\pi\right)} - \frac{1}{9\cos^2\left(\frac{\pi}{2}\pi - \frac{\varepsilon\pi}{6}\right)}$$

$$= \frac{1}{\left(\frac{\varepsilon\pi}{2}\right)^2\left(1 - \frac{1}{3}\frac{\varepsilon\pi^2}{4} + \cdots\right)} - \frac{1}{\left(\frac{\varepsilon\pi}{2}\right)^2\left(1 - \frac{1}{3}\frac{\varepsilon^2\pi^2}{36} + \cdots\right)}$$

$$= \frac{1}{3}\cdot\frac{8}{9} = \frac{8}{27} \quad ; \qquad \boxed{\frac{W_1}{P_E l} = \frac{1}{18\left(\frac{P}{P_E}\right)}\left(\frac{F_1}{P_E}\right)^2}$$

75

$$\frac{W_1}{P_E l} = \frac{3\left(\frac{F_1}{P_E}\right)^2}{16\left(\frac{P}{P_E}\right)} \left\{ \frac{8}{9} + \left(\tan^2 \frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\tan^2 \frac{\pi}{6}\sqrt{\frac{P}{P_E}} \right) - \frac{2}{\pi\sqrt{\frac{P}{P_E}}} \left(\tan \frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\tan \frac{\pi}{6}\sqrt{\frac{P}{P_E}} \right) \right\}$$

since $\left(\frac{F}{P_E}\right) / (P/P_E) = \left(\frac{\xi}{l}\right) / \sqrt{\beta}$ see p. 31. Q

$$\therefore \frac{W_1}{P_E l} = \frac{3}{16}\, \xi^2 \frac{P}{P_E} \frac{1}{\beta} \left[\frac{8}{9} + \left(\tan^2 \frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\tan^2 \frac{\pi}{6}\sqrt{\frac{P}{P_E}} \right) - \frac{2}{\pi\sqrt{\frac{P}{P_E}}} \left(\tan \frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\tan \frac{\pi}{6}\sqrt{\frac{P}{P_E}} \right) \right]$$

$$\frac{W_2}{P_E l} = 2\left(13.500 - 1880\,\xi + 70500\,\xi^2 \right) \xi^2$$

$$\frac{W_3}{P_E l} = \frac{\pi^2}{2} \left(\frac{\delta}{l}\right)^2 \left(\frac{P}{P_E}\right)^2$$

$\frac{8}{9} = 0.888889$; $\frac{2}{\pi} = 0.636620$; $\frac{3}{16} = 0.1875$ 76.

①	②	③	④	⑤	⑥	⑦	⑧	⑨
$\sqrt{\frac{P}{P_E}}$	Q	$\frac{3}{16}\frac{Q}{\beta}$	$10^4 \times \xi_1^2$	$10^4 \times \xi_2^2$	$\left[\frac{W_1}{P_E l}\right]_1$	$\left[\frac{W_1}{P_E l}\right]_2$	$\left[\frac{W_2}{P_E l}\right]_1$	$\left[\frac{W_2}{P_E l}\right]_2$
3.0		0.50000		4.00000	0	18.0000	0	32.8000
2.9	0.28895	0.47658	0.001380	3.85274	0.00553	15.4419	0.09521	28.9759
2.8	0.28159	0.45316	0.005509	3.70863	0.01957	13.1759	0.13380	25.5241
2.7	0.27787	0.43551	0.011365	3.56757	0.03608	11.3266	0.26132	22.4179
2.6	0.27715	0.42194	0.021951	3.42931	0.06261	9.78146	0.47719	19.6297
2.5	0.27914	0.41153	0.034273	3.29375	0.08815	8.47172	0.70336	17.1317
2.4	0.28487	0.40511	0.049346	3.16079	0.11515	7.37548	0.95452	14.9175
2.3	0.29512	0.40293	0.067190	3.03035	0.14322	6.45919	1.22293	12.9524
2.2	0.31134	0.40573	0.087853	2.90225	0.17252	5.69924	1.50177	11.2213
2.1	0.33569	0.41447	0.11142	2.77626	0.20365	5.07448	1.78500	9.70497
2.0	0.37175	0.43077	0.13792	2.65244	0.23765	4.57037	2.06589	8.38930
1.9	0.42544	0.45713	0.16747	2.53055	0.27637	4.17601	2.34024	7.25711
1.8	0.50178	0.49697	0.20020	2.41044	0.32236	3.88124	2.60242	6.29361
1.7	0.63412	0.55539	0.24315	2.27075	0.39028	3.64474	2.89052	5.35466
1.6	0.84332	0.64022	0.27609	2.17432	0.45250	3.56362	3.07462	4.81516
1.5	1.21190	0.76355	0.31979	2.05779	0.54940	3.53526	3.27665	4.27535
1.4	1.93090	0.94425	0.36793	1.94165	0.68094	3.59347	3.45146	3.85274
1.3	3.57453	1.21305	0.42121	1.82517	0.86350	3.74169	3.59559	3.53667
1.2	8.54119	1.62155	0.48094	1.70694	1.12300	3.98576	3.70597	3.31740
1.1	36.92677	2.25931	0.54916	1.58496	1.50128	4.33292	3.77814	3.18789
1.0								
0.9	36.37818	4.05854	0.73481	1.30597	2.41563	4.49327	3.76935	3.19344
0.8	8.00806	5.13372	0.95164	1.04956	3.12669	3.44841	3.55776	3.44081

Incorrect,

$\sqrt{\frac{P}{P_E}}$	$\left(\frac{W_g}{P_E l}\right)$	$\left[\frac{W}{P_E l}\right]_1$	$\left[\frac{W}{P_E l}\right]_2$		$\left[\frac{W}{P_E l}\right]_1$/π值	$\left[\frac{W}{P_E l}\right]_1$/π值
3.0	10.12500	10.1250			40.5000	91.30000
2.9	8.84101	8.94175	53.2588		35.46479	79.78185
2.8	7.68320	7.83657	46.3832		30.88617	69.43280
2.7	6.64301	6.94041	40.3875		26.81945	60.31655
2.6	5.71220	6.25200	35.1234		23.38860	52.25976
2.5	4.88281	5.67432	30.4912		20.32276	45.13967
2.4	4.1472	5.21687	26.4402		17.65847	38.88178
2.3	3.49801	4.86416	22.9096		15.35820	33.40364
2.2	2.92820	4.60249	19.8487		13.38709	28.63334
2.1	2.43101	4.41966	17.2105		11.71270	24.50350
2.0	2.00000	4.30354	14.9597		10.30354	20.95767
1.9	1.62901	4.24565	13.0621		9.13019	17.94917
1.8	1.31220	4.23698	11.4871		8.17358	15.42365
1.7	1.04401	4.32481	10.0434		7.45685	13.17545
1.6	0.81920	4.34632	9.19798		6.80372	11.65558
1.5	0.63281	4.45886	8.44342		6.35730	10.34186
1.4	0.48020	4.61260	7.92641		6.05320	9.36701
1.3	0.35701	4.81610	7.63537		5.88714	8.70641
1.2	0.25920	5.08817	7.56236		5.86577	8.33996
1.1	0.18301	5.46243	7.70382		6.01147	8.25286
1.0	0.12500					
0.9	0.08201	6.26699	7.51872		6.51303	7.81476
0.8	0.05120	6.73565	6.94042		6.88725	7.09402

Three Wave Euler-Buckling

$$w = a_3 \sin \frac{3\pi x}{l}$$

$$\frac{dw}{dx} = \frac{3\pi}{l} a_3 \cos \frac{3\pi x}{l}$$

$$\frac{d^2w}{dx^2} = -\left(\frac{3\pi}{l}\right)^2 a_3 \sin \frac{3\pi x}{l}$$

$$\therefore \quad a_3^2 = \frac{\varepsilon 4}{l\left(\frac{3\pi}{l}\right)^2}$$

$$\varepsilon^* = \frac{1}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx = \frac{1}{2}\left(\frac{3\pi}{l}\right)^2 a_3^2 \frac{l}{2}$$

Bending Strain energy $= \frac{EI}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2}\left(\frac{3\pi}{l}\right)^4 a_3^2 \frac{l}{2}$.

Compression Strain energy $= \frac{Pl}{AE}\frac{1}{2}P = \frac{l}{2}\frac{P_E^2}{AE}81$

Total Strain energy $= W_E = \frac{l}{2}\cdot\left\{\frac{81P_E^2}{AE} + \frac{EI}{2}\left(\frac{3\pi}{l}\right)^2\frac{4\varepsilon}{l}\right\}$

$$= \frac{l}{2}\left\{\frac{81P_E}{AE} + \frac{9}{2}\frac{4\varepsilon}{l}\right\}P_E$$

$$\therefore \quad \frac{W_E}{lP_E} = \frac{1}{2}\left\{\frac{9P_E}{AE} + \frac{9}{2}\frac{4\varepsilon}{l}\right\}, \quad \text{now} \quad \frac{P_E}{AE} = \frac{EI\pi^2}{AEl^2} = \pi^2\left(\frac{R}{l}\right)^2$$

$$= \frac{1}{2}\pi^2\left(\frac{R}{l}\right)^2\left\{81 + 18\left(\frac{\frac{\varepsilon}{l}}{\pi^2\left(\frac{R}{l}\right)^2}\right)\right\}$$

$$\boxed{\frac{W_E}{lP_E}\times 10^4 = \left\{\frac{81}{8} + \frac{9}{4}\left[\frac{\frac{\varepsilon^*}{l}}{\pi^2\left(\frac{R}{l}\right)^2}\right]\right\}}$$

$$\frac{\varepsilon^*}{l} = \frac{\varepsilon}{l} - \frac{Pl}{AE}$$

$$=$$

Bending strain energy $= \frac{EI}{2}\frac{l}{2}\left(\frac{\pi}{l}\right)^4 \sum_{n=1,2,3}^{\infty} n^4 a_n^2$

$$= \frac{P_E}{2}\frac{l}{2}\pi^2 \sum_{n=1,2,3}^{\infty} n^4 \left(\frac{a_n}{l}\right)^2 = \frac{P_E l}{\pi^2} \sum_{n=1,2,3}^{\infty} \frac{\sin^2\frac{n\pi}{3}\left[\left\{\left(\frac{F_1}{P_E}\right)^2 + \left(\frac{F_2}{P_E}\right)^2\right\} - 2(-1)^n\left(\frac{F_1F_2}{P_E}\right)^2\right]}{\left(\frac{P}{P_E} - n^2\right)^2}$$

$$\boxed{\frac{W_1}{P_E l} = \left[b\left\{\left(\frac{F_1}{P_E}\right)^2 + \left(\frac{F_2}{P_E}\right)^2\right\} + d\left(\frac{F_1F_2}{P_E}\right)^2\right]}$$

$$b = \frac{1}{\pi^2}\sum_{n=1,2,3}^{\infty} \frac{\sin^2\frac{n\pi}{3}}{\left(\frac{P}{P_E} - n^2\right)^2} = \frac{3}{4\pi^2}\left[\sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{P_E} - n^2\right)^2} - \frac{1}{81}\sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{9P_E} - n^2\right)^2}\right]$$

$$c = \frac{3}{2\pi^2}\left[\sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{P_E} - n^2\right)^2} - \frac{1}{81}\sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{9P_E} - n^2\right)^2}\right.$$

$$\left. - 2\left\{\frac{1}{16}\sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{4P_E} - n^2\right)^2} - \frac{1}{1296}\sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{36P_E} - n^2\right)^2}\right\}\right]$$

But $\sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{P_E} - n^2\right)^2} = -\frac{\partial}{\partial\left(\frac{P}{P_E}\right)} \sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{P_E} - n^2\right)} = -\frac{1}{2}\frac{1}{\sqrt{\frac{P}{P_E}}}\frac{\partial}{\partial\sqrt{\frac{P}{P_E}}} \sum_{n=1,2,3}^{\infty} \frac{1}{\left(\frac{P}{P_E} - n^2\right)}$

$$= -\frac{1}{2}\frac{1}{\sqrt{\frac{P}{P_E}}}\left[-\frac{\pi}{2\frac{P}{P_E}}\cot\pi\sqrt{\frac{P}{P_E}} - \frac{\pi^2}{2\sqrt{\frac{P}{P_E}}}\csc^2\pi\sqrt{\frac{P}{P_E}} + \frac{1}{\frac{P}{P_E}\sqrt{\frac{P}{P_E}}}\right]$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)}\left[\frac{\pi^2}{4}\csc^2\pi\sqrt{\frac{P}{P_E}} + \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{2\frac{P}{P_E}}\right]$$

$$b = \frac{3}{4\pi^2}\left[\frac{1}{\left(\frac{P}{P_E}\right)}\left\{\frac{\pi^2}{4}\csc^2\pi\sqrt{\frac{P}{P_E}} + \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\cot\pi\sqrt{\frac{P}{P_E}}\right\}\right.$$

$$\left. - \frac{1}{\left(\frac{P}{P_E}\right)}\left\{\frac{\pi^2}{4}\frac{1}{9}\csc^2\frac{\pi}{3}\sqrt{\frac{P}{P_E}} + \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\frac{1}{3}\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right\}\right]$$

$$= \frac{1}{\left(\frac{P}{P_E}\right)}\left[\frac{1}{6} + \frac{3}{16}\left(\cot^2\pi\sqrt{\frac{P}{P_E}} - \frac{1}{9}\cot^2\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right) + \frac{3}{16\pi\sqrt{\frac{P}{P_E}}}\left(\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{3}\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right)\right]$$

$$\boxed{b = \frac{1}{\left(\frac{P}{P_E}\right)}\left[\frac{1}{6} + \frac{3}{16}\left(\cot^2\pi\sqrt{\frac{P}{P_E}} - \frac{1}{9}\cot^2\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right) + \frac{3}{16\pi\sqrt{\frac{P}{P_E}}}\left(\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{3}\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right)\right]}$$

$$d = \frac{1}{\left(\frac{P}{P_E}\right)}\left[\frac{1}{6} + \frac{3}{16}\left(\cot^2\pi\sqrt{\frac{P}{P_E}} - \frac{1}{9}\cot^2\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right) + \frac{3}{16\pi\sqrt{\frac{P}{P_E}}}\left(\cot\pi\sqrt{\frac{P}{P_E}} - \frac{1}{3}\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}\right)\right.$$

$$\left. - \frac{1}{2}\cdot\frac{1}{6} - \frac{1}{2}\frac{3}{16}\left(\cot^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{9}\cot^2\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right) - \frac{3}{16\pi\sqrt{\frac{P}{P_E}}}\left(\cot\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{3}\cot\frac{\pi}{6}\sqrt{\frac{P}{P_E}}\right)\right]$$

$$\boxed{d = -\frac{1}{\left(\frac{P}{P_E}\right)}\left[\frac{3}{16}\left(\frac{\cot\pi\sqrt{\frac{P}{P_E}}}{\sin\pi\sqrt{\frac{P}{P_E}}} - \frac{1}{9}\frac{\cot\frac{\pi}{3}\sqrt{\frac{P}{P_E}}}{\sin\frac{\pi}{3}\sqrt{\frac{P}{P_E}}}\right) + \frac{3}{16\pi\sqrt{\frac{P}{P_E}}}\left(\frac{1}{\sin\pi\sqrt{\frac{P}{P_E}}} - \frac{1}{3}\frac{1}{\sin\frac{\pi}{3}\sqrt{\frac{P}{P_E}}}\right)\right]}$$

$$\boxed{\frac{M_2}{P_E l} = \xi_1^2(13.500 - 1880\xi_1 + 70500\xi_1^2) + \xi_2^2(13.500 - 1880\xi_2 + 70500\xi_2^2)}$$

$$\boxed{\frac{M_3}{P_E l} = 10^{-4}\frac{1}{8}\left(\frac{P}{P_E}\right)^2}$$

when $\sqrt{\frac{P}{P_E}} = 3$,

$$b = \frac{1}{54}\frac{1}{3}$$

$$d = -\frac{1}{108}\frac{1}{3}$$

$\frac{1}{\pi} = 0.31831$ $\frac{1}{3\pi} = 0.10610$

P/P_E	b	d	$10^4\left(\frac{M_1}{P_E l}\right)_1$	$10^4\left(\frac{M_1}{P_E l}\right)_2$	$10^4\left(\frac{M_2}{P_E l}\right)_{1,1}$	$10^4\left(\frac{M_2}{P_E l}\right)_{1,2}$	$10^4\left(\frac{M_2}{P_E l}\right)_{2,1}$	$10^4\left(\frac{M_2}{P_E l}\right)_{2,2}$	$10^4\left(\frac{M}{P_E l}\right)_1$	$10^4\left(\frac{M}{P_E l}\right)_2$
9.00	0.006173	-0.003086	50.7725	331.09	32.03579	1.64422	71.73497	43.88435	124.95251	487.20932
7.84	0.008273	-0.004494	40.0510	171.958	22.63768	1.81576	39.87262	21.20237	95.28724	263.76579
6.76	0.013545	-0.008329	34.7427	100.365	15.21587	1.86714	21.58261	8.11763	74.67451	152.91404
5.76	0.028904	-0.021682	33.1722	61.466	9.53121	1.52963	10.94399	1.78040	60.82184	90.77917
5.29	0.050739	-0.041987	34.9170	49.151	7.24987	1.15637	7.57575	0.35000	57.3153	71.06880

81

If there is a spring of a constant k

$$P/P_E = k\,\varepsilon_s \qquad \varepsilon_s = \text{Spring deflection}$$

or $$\varepsilon_s = \frac{1}{k}\frac{P}{P_E}$$

Energy Stored in the Spring $= \frac{1}{2}\,\varepsilon_s \frac{P}{P_E} P_E = \frac{1}{2}\frac{1}{k}\left(\frac{P}{P_E}\right)^2 P_E$

$$\frac{\text{Total deflection of Testing Machine}}{l\,\pi^2\left(\frac{r}{l}\right)^2} = \frac{\varepsilon_{TOT}/l}{\pi^2\left(\frac{r}{l}\right)^2} + \frac{1}{k\,l\,\pi^2\left(\frac{r}{l}\right)^2}\left(\frac{P}{P_E}\right)$$

Energy Stored in the Spring $= W_4$

$$\frac{W_4}{P_E\,l} = \frac{1}{2}\frac{1}{k\,l\,P_E}\left(\frac{P}{P_E}\right)^2 = \frac{1}{2}\frac{\pi^2\left(\frac{r}{l}\right)^2}{k\,l\,\pi^2\left(\frac{r}{l}\right)^2}\left(\frac{P}{P_E}\right)^2$$

Let $$\boxed{\frac{1}{k\,l\,\pi^2\left(\frac{r}{l}\right)^2} = 2};$$

$$\frac{\text{Total deflection}}{l\,\pi^2\left(\frac{r}{l}\right)^2} = \frac{\varepsilon_{TOT}/l}{\pi^2\left(\frac{r}{l}\right)^2} + 2\left(\frac{P}{P_E}\right)$$

$$\boxed{\frac{W_4}{P_E\,l} = \frac{1}{4}\left(\frac{P}{P_E}\right)^2 \times 10^{-4}}$$

$\sqrt{\frac{P}{P_E}}$	$(W/P_E l)10^4$	$\frac{\varepsilon_H}{l}/\pi^2(\frac{h}{l})^2$	$(W/P_E l)10^4$	$(\frac{\varepsilon_H}{l})/\pi^2(\frac{h}{l})^2$			
3.00	30.3750	27.0000	—				
2.9	26.62377	25.2496					
2.8	23.20297	23.5960					
2.7	20.22643	22.0404					
2.6	17.67640	20.5804					
2.5	15.43994	19.2164					
2.4	13.51127	17.9476					
2.3	11.86018	16.7764					
2.2	10.45889	15.7020					
2.1	9.28168	14.7236					
2.0	8.30354	13.8436					
1.9	7.50367	13.0644					
1.8	6.86138	12.3848					
1.7	6.41283	11.7992					
1.6	5.98472	11.3412					
1.5	5.72448	10.8836					
1.4	5.57300	10.7436					
1.3	5.53012	10.6304					
1.2	5.60657	10.6612					
1.1	5.87845	10.8640					
1.0							
0.9	6.43101	12.0916	7.73274	19.6024			
0.8	6.83805	14.4224	7.04282	15.7100			
0.7							

Section 4

Buckling of Column with One Non-linear Support

Buckling of Column with one non-linear support.

Let $w = \sum_{n=1,3,5}^{\infty} a_n \sin \frac{n\pi x}{l}$

The lowering of the potential of $P = -\frac{1}{2}P\int_0^l \left(\frac{dw}{dx}\right)^2 dx$

$$= -\frac{P}{2}\frac{l}{2}\sum_{n=1,3,5}^{\infty}\left(\frac{n\pi}{l}\right)^2 a_n^2$$

The increase in bending strain energy $= \frac{EI}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2}\frac{l}{2}\sum_{n=1,3,5}^{\infty}\left(\frac{n\pi}{l}\right)^4 a_n^2$

Work done on the supporting spring $= W_2$

Total potential of the system

$$\frac{l}{4}\left(\frac{\pi}{l}\right)^2\left\{P_E\sum_{n=1,3,5}^{\infty} n^2\left[n^2 - \frac{P}{P_E}\right]a_n^2\right\} + W_2$$

$$\frac{l}{2}\left(\frac{\pi}{l}\right)^2 P_E \qquad n^2\left[n^2 - \frac{P}{P_E}\right]a_n + \sin\frac{n\pi}{2} F = 0$$

or

$$\frac{a_n}{l} = \frac{2}{\pi^2}\quad\frac{\sin\frac{n\pi}{2}\left(\frac{F}{P_E}\right)}{n^2\left[\frac{P}{P_E} - n^2\right]}$$

or

$$\boxed{\frac{a_n}{l} = \frac{2(-)^{\frac{n-1}{2}}}{\pi^2}\frac{\left(\frac{F}{P_E}\right)}{n^2\left[\frac{P}{P_E} - n^2\right]}}$$

$$\xi = \frac{f}{l} = \sum_{n=1,3,5}^{\infty} \sin\frac{n\pi}{2}\cdot\frac{q_n}{l} = \frac{2}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{\frac{F}{P_E}}{n^2\left[\frac{P}{P_E}-n^2\right]}$$

$$\therefore \quad \xi = \frac{2\frac{F}{P_E}}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]}$$

$$= \frac{2\frac{F}{P_E}}{\pi^2\frac{P}{P_E}}\sum_{n=1,3,5}^{\infty}\left\{\frac{1}{n^2}+\frac{1}{\frac{P}{P_E}-n^2}\right\}$$

$$= \frac{2\frac{F}{P_E}}{\pi^2\frac{P}{P_E}}\left\{\frac{\pi^2}{8}-\frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}$$

$$\therefore \quad \boxed{\xi = \frac{\frac{F}{P_E}}{\frac{P}{P_E}}\left\{\frac{1}{4}-\frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}$$

The shortening due to deflection of the column from straight position $= \varepsilon_2$

$$\frac{\varepsilon_2}{l} = \frac{\pi^2}{4}\sum_{n=1,3,5}^{\infty} n^2\left(\frac{q_n}{l}\right)^2 = \frac{1}{\pi^2}\left(\frac{F}{P_E}\right)^2\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left[\frac{P}{P_E}-n^2\right]^2}$$

$$= \frac{\left(\frac{F}{P_E}\right)^2}{\pi^2}\frac{1}{\left(\frac{P}{P_E}\right)^2}\sum_{n=1,3,5}^{\infty}\left\{\frac{1}{n^2}+\frac{1}{\frac{P}{P_E}-n^2}-\frac{P}{P_E}\frac{\partial}{\partial\left(\frac{P}{P_E}\right)}\left(\frac{1}{\frac{P}{P_E}-n^2}\right)\right\}$$

$$= \frac{\left(\frac{F}{P_E}\right)^2}{\left(\frac{P}{P_E}\right)^2}\frac{1}{\pi^2}\left\{\frac{\pi^2}{8}-\frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}-\frac{1}{2}\sqrt{\frac{P}{P_E}}\left[\frac{\pi}{4\frac{P}{P_E}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{\pi^2}{8\sqrt{\frac{P}{P_E}}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]\right\}$$

$$\frac{\varepsilon_2}{l} = \frac{\left(\frac{F}{P_E}\right)^2}{\left(\frac{P}{P_E}\right)^2}\frac{1}{\pi^2}\left\{\frac{\pi^2}{8} + \frac{\pi^2}{16}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3\pi}{8\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}$$

$$\boxed{\frac{\varepsilon_2}{l} = \frac{\left(\frac{F}{P_E}\right)^2}{\left(\frac{P}{P_E}\right)^2}\left\{\frac{3}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}$$

$$\boxed{\frac{\varepsilon_2}{l} = \xi^2\frac{\left\{\frac{3}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}{\left\{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}^2}}$$

The shortening due to compression $= \frac{Pl}{EA} = \varepsilon_1$

$$\therefore \frac{\varepsilon_1}{l} = \frac{P}{EA} = \frac{P_E}{EA}\frac{P}{P_E} = \frac{\pi^2 I}{l^2 A}\frac{P}{P_E} = \boxed{\pi^2\left(\frac{i}{l}\right)^2\frac{P}{P_E} = \frac{\varepsilon_1}{l}}$$

where i = radius of gyration of the column section.

The strain energy of bending $= \frac{EIl}{4}\frac{\pi^4}{l^2}\sum_{n=1,3,5}^{\infty} n^4\left(\frac{a_n}{l}\right)^2$

$$= \frac{P_E l}{4}\pi^2\sum_{n=1,3,5}^{\infty} n^4\left(\frac{a_n}{l}\right)^2 = W_1$$

$$\frac{W_1}{P_E l} = \frac{1}{\pi^2}\left(\frac{F}{P_E}\right)^2\sum_{n=1,3,5}^{\infty}\frac{1}{\left[\frac{P}{P_E} - n^2\right]^2} = -\frac{1}{\pi^2}\frac{\partial}{\partial\left(\frac{P}{P_E}\right)}\sum_{n=1,3,5}^{\infty}\frac{1}{\frac{P}{P_E} - n^2}$$

$$= -\frac{1}{2}\frac{1}{\pi^2\sqrt{\frac{P}{P_E}}}\left[\frac{\pi}{4\left(\frac{P}{P_E}\right)}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{\pi^2}{8\sqrt{\frac{P}{P_E}}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]\left(\frac{F}{P_E}\right)^2$$

$$\frac{W_1}{P_E l} = \frac{1}{\frac{P}{P_E}}\left[\frac{1}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]\left(\frac{F}{P_E}\right)^2$$

$$\frac{W_1}{P_E l} = \left(\frac{P}{P_E}\right)\xi^2 \frac{\left\{\frac{1}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}{\left\{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}^2}$$

The strain energy stored in the support $= \int_0^\delta F\,d\delta = W_2$

$$\frac{W_2}{P_E l} = \int_0^\xi \left(\frac{F}{P_E}\right) d\xi$$

The strain energy of compression $= \frac{1}{2}P\,\frac{Pl}{EA} = W_3$

$$\frac{W_3}{P_E l} = \frac{1}{2}\frac{P_E}{EA}\left(\frac{P}{P_E}\right)^2 = \frac{1}{2}\frac{\frac{EI\pi^2}{l^2}}{EA}\left(\frac{P}{P_E}\right)^2$$

$$\frac{W_3}{P_E l} = \frac{\pi^2}{2}\left(\frac{i}{l}\right)^2\left(\frac{P}{P_E}\right)^2$$

If there is a spring between the end plate of the test machine and the column, and let

$$P = \varepsilon_s \cdot K.$$

then

$$\frac{\varepsilon_s}{l} = \frac{P}{Kl} = \left(\frac{P_E}{Kl}\right)\left(\frac{P}{P_E}\right)$$

Strain energy stored in the spring

$$W_4 = \frac{1}{2} P \varepsilon_s .$$

$$\therefore \quad \boxed{\frac{W_4}{P_E l} = \frac{1}{2}\left(\frac{P}{P_E}\right)\left(\frac{\varepsilon_s}{l}\right) = \frac{1}{2}\left(\frac{P_E}{kl}\right)\left(\frac{P}{P_E}\right)^2}$$

Summary:

$$\boxed{\xi = \frac{\frac{F}{P_E}}{\frac{P}{P_E}}\left(\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)}$$

ψ

$$\boxed{\frac{\varepsilon}{l} = \left\{\pi^2\left(\frac{i}{l}\right)^2 + \alpha\right\}\frac{P}{P_E} + \xi^2 \frac{\left\{\frac{3}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}{\left\{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}^2}} \qquad \alpha = \frac{P_E}{kl}$$

$$\boxed{\frac{W}{P_E l} = \left\{\frac{\pi^2}{2}\left(\frac{i}{l}\right)^2 + \frac{\alpha}{2}\right\}\left(\frac{P}{P_E}\right)^2 + \int_0^{\xi}\left(\frac{F}{P_E}\right)d\xi + \left(\frac{P}{P_E}\right)\xi^2 \frac{\left\{\frac{1}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}{\left\{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}^2}}$$

Λ

$$\boxed{\frac{\frac{P}{P_E}}{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}} = \frac{\frac{F}{P_E}}{\xi}}$$

$$\frac{1}{2\pi} = 0.159155$$

H

①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩
$\sqrt{\frac{p}{p_E}}$	$\frac{p}{p_E}$	$\tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}$	$\frac{1}{4}-\frac{1}{2\pi\sqrt{\frac{p}{p_E}}}\tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}$	②/④	$\frac{3}{4}$④$+\frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{p}{p_E}}$	④^2	⑥/⑦	⑥$-\frac{1}{2}$④	⑨/⑦
3.0	9.00	∞	−∞	−0	∞	∞	22.2066	∞	22.2066
2.9	8.41	6.3138	−0.096508	−87.143	2.41912	0.0093138	259.735	2.46737	264.916
2.8	7.84	3.0777	+0.075060	104.450	0.648310	0.0056340	115.071	0.610780	108.410
2.7	7.29	1.9626	+0.134312	54.2766	0.341471	0.0180397	18.9289	0.274315	15.2062
2.6	6.76	1.3764	+0.165746	40.7853	0.242714	0.0274717	8.83506	0.159841	5.81839
2.5	6.25	1.0000	+0.186338	33.5412	0.202254	0.0347219	5.82497	0.109085	3.14168
2.4	5.76	0.72654	+0.201820	28.5403	0.184356	0.0407313	4.52615	0.083446	2.04869
2.3	5.29	0.50953	+0.214742	24.6342	0.177283	0.0461141	3.84444	0.069912	1.51607
2.2	4.84	0.32492	+0.221494	21.3692	0.176469	0.0512995	3.43998	0.063222	1.23241
2.1	4.41	0.15838	+0.237997	18.5296	0.180066	0.0566426	3.17899	0.061068	1.07813
2.0	4.00	0	+0.250000	16.0000	0.187500	0.0625000	3.00000	0.062500	1.00000
1.9	3.61	−0.15838	+0.263267	13.7123	0.199018	0.0693095	2.87144	0.067385	0.97223
1.8	3.24	−0.32492	+0.278729	11.6242	0.215645	0.0776899	2.77571	0.076281	0.98187
1.7	2.89	−0.50953	+0.297702	9.70719	0.239503	0.0886265	2.70239	0.090652	1.02285
1.6	2.56	−0.72654	+0.322270	7.94365	0.274694	0.103858	2.64490	0.113559	1.09341
1.5	2.25	−1.0000	+0.356103	6.31840	0.329577	0.126809	2.59900	0.151526	1.19492
1.4	1.96	−1.3764	+0.406472	4.82198	0.423259	0.165219	2.56181	0.220023	1.33171
1.3	1.69	−1.9626	+0.490275	3.44705	0.608444	0.240370	2.53128	0.363307	1.51145
1.2	1.44	−3.0777	+0.658193	2.18781	1.085660	0.433218	2.50604	0.756564	1.74638
1.1	1.21	−6.3138	+1.163521	1.03995	3.364145	1.353781	2.48500	2.782385	2.05527
1.0	1.00	−∞ (+∞)	+∞ (−∞)	0	∞	∞	2.46740	∞	2.46740
0.9	0.81	+6.3138	−0.866525	−0.93477	1.841610	0.750866	2.45265	2.274873	3.02967
0.8	0.64	+3.0777	−0.362289	−1.76655	0.320298	0.131253	2.44031	0.501443	3.82043

H

⑪	⑫	⑬	⑭	⑮	⑯	⑰	⑱
$\frac{P}{P_E}$	⑩⑪	$0.01-\frac{13.7123-H}{143.2174}$	$⑬^{\frac{1}{2}}$	ξ_1	ξ_2	ξ_1^2	ξ_2^2
7.84	849.930						
7.29	110.853						
6.76	39.3323						
6.25	19.6355						
5.76	11.8005						
5.29	8.02001						
4.84	5.91486						
4.41	4.75455						
4.00	4.00000						
3.61	3.50975	0.010000	0.10000	0	0.20000	0	0.0400000
3.24	3.18126	0.0085420	0.092423	0.007577	0.192423	0.00005741	0.0370266
2.89	2.95604	0.0072038	0.084875	0.015125	0.184875	0.00022877	0.0341788
2.56	2.79913	0.0059719	0.077278	0.022722	0.177278	0.00051629	0.0314275
2.25	2.68157	0.0048373	0.069550	0.030450	0.169550	0.00092720	0.0287472
1.96	2.61015	0.0037925	0.061583	0.038417	0.161583	0.0014759	0.0261091
1.69	2.55435	0.0028324	0.053220	0.046780	0.153220	0.0021884	0.0234764
1.44	2.51479	0.0019532	0.044195	0.055805	0.144195	0.0031142	0.0207922
1.21	2.48688	0.0011517	0.033937	0.066063	0.133937	0.0043643	0.0179391
1.00	2.46740	0.0004255	0.020868	0.079132	0.120868	0.0062619	0.0146091
0.81	2.45403						
0.64	2.44508						

$$\frac{\frac{E}{P_E}}{\xi} = K\left(1 - 20.8889\,\xi + 104.444\,\xi^2\right), \quad \text{for } \left(\frac{E}{P_E}\right)_{MAX.} \text{ at } \xi = 0.1$$

$$K = 13.7123\,; \qquad \frac{\frac{E}{P_E}}{\xi} = 13.7123 - 286.4347\xi + 143.2174\,\xi^2$$

$$\xi = 0.1 \pm \sqrt{0.01 - \frac{13.7123 - H}{143.2174}}$$

$$\Phi(\xi) = \int_0^{\xi} \left(\frac{F}{P_E}\right) d\xi = K\xi^2 \left[\frac{1}{2} - \frac{1}{3} 20.88889\xi + \frac{1}{4} 104.4444\xi^2 \right]$$

$$= K\xi^2 \left[0.500000 - 6.96296296\xi + 26.111111\xi^2 \right]$$

$$\frac{\xi}{\pi \frac{i}{l}} = \frac{1}{\pi}\left(\frac{\xi}{i}\right) \qquad \text{Let} \quad \frac{i}{l} = \frac{1}{10\pi} \; ; \qquad \frac{\xi}{\pi \frac{i}{l}} = 10\left(\frac{\xi}{l}\right) = \eta$$

$\left(\frac{F}{P_E}\right)$ max. at $\eta = 1.00$

(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)
$\frac{P}{P_E}$	ϕ_1	ϕ_2	ψ_1	ψ_2	Λ_1	Λ_2	
3.61	0	0.0832884	0	0.114858	0	0.140390	
3.24	0.0003533	0.0644652	0.0001594	0.102775	0.0001826	0.117791	
2.89	0.0012569	0.0492899	0.0006182	0.092364	0.0006763	0.101034	
2.56	0.0025151	0.0371592	0.0013655	0.083123	0.0014452	0.0879197	
2.25	0.0039692	0.0276137	0.0024098	0.074714	0.0024928	0.0772887	
1.96	0.0054853	0.0202779	0.0037810	0.066887	0.0038523	0.0681487	
1.69	0.0069441	0.0148495	0.0055395	0.059425	0.0055899	0.0599670	
1.44	0.0082310	0.0110860	0.0078043	0.052106	0.0078316	0.0522880	
1.21	0.0092137	0.0088092	0.0108453	0.044579	0.0108535	0.0446124	
1.00	0.0096607	0.0079849	0.0154506	0.036046	0.0154506	0.0360465	

$$\frac{\frac{\varepsilon}{l}}{\left(\pi^2 \left(\frac{i}{l}\right)^2 + \alpha\right)} = u \; ; \qquad \frac{\frac{W}{P_E l}}{\left(\pi^2 \left(\frac{i}{l}\right)^2 + \alpha\right)} - \frac{1}{2}\left(\frac{P}{P_E}\right)^{*2} = \Theta$$

$$\frac{\alpha}{\pi^2 \left(\frac{i}{l}\right)^2} = \beta$$

①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩
	$\beta=0$		$\beta=3$						
$\frac{P}{P_E}$	x	Θ	x	Θ	Potential				
3.61	3.61000	0	3.61000	0	−6.51605				
3.24	3.25594	+0.0018	3.24399	+0.0005	−5.2469				
2.89	2.95182	+0.0128	2.90546	+0.0035	−4.1613				
2.56	2.69655	+0.0371	2.59414	+0.0110	−3.2304				
2.25	2.49098	+0.0750	2.31025	+0.0242	−2.4272				
1.96	2.33810	+0.1212	2.05453	+0.0437	−1.72112				
1.69	2.24395	+0.1638	1.82849	+0.0697	−1.11082				
1.44	2.22043	+0.1779	1.63511	+0.1016	−0.5544				
1.21	2.29453	+0.1063	1.48113	+0.1369	−0.0376				
1.00	2.54506	−0.2275	1.38627	+0.1669	[illegible]				
1.00	4.6046	−5.6980	1.90115	−0.2064	+0.2976				
1.21	5.6679	−9.9883	2.32448		−1.7439				
1.44	6.6506	−14.7410	2.74265		−2.2026				
1.69	7.6325		3.17563						
1.96	8.6487		3.63218						
2.25	9.7214		4.11785						
2.56	10.8723		4.63808						
2.89	12.1264		5.19910						
3.24	13.5175		5.80938						
3.61	15.0958	[illegible]	6.48145						

$\beta = 0$

$\frac{P}{P_E}$

5

4

3

2

1

1 2 3 4 5 6 7 $\varkappa$

0.2

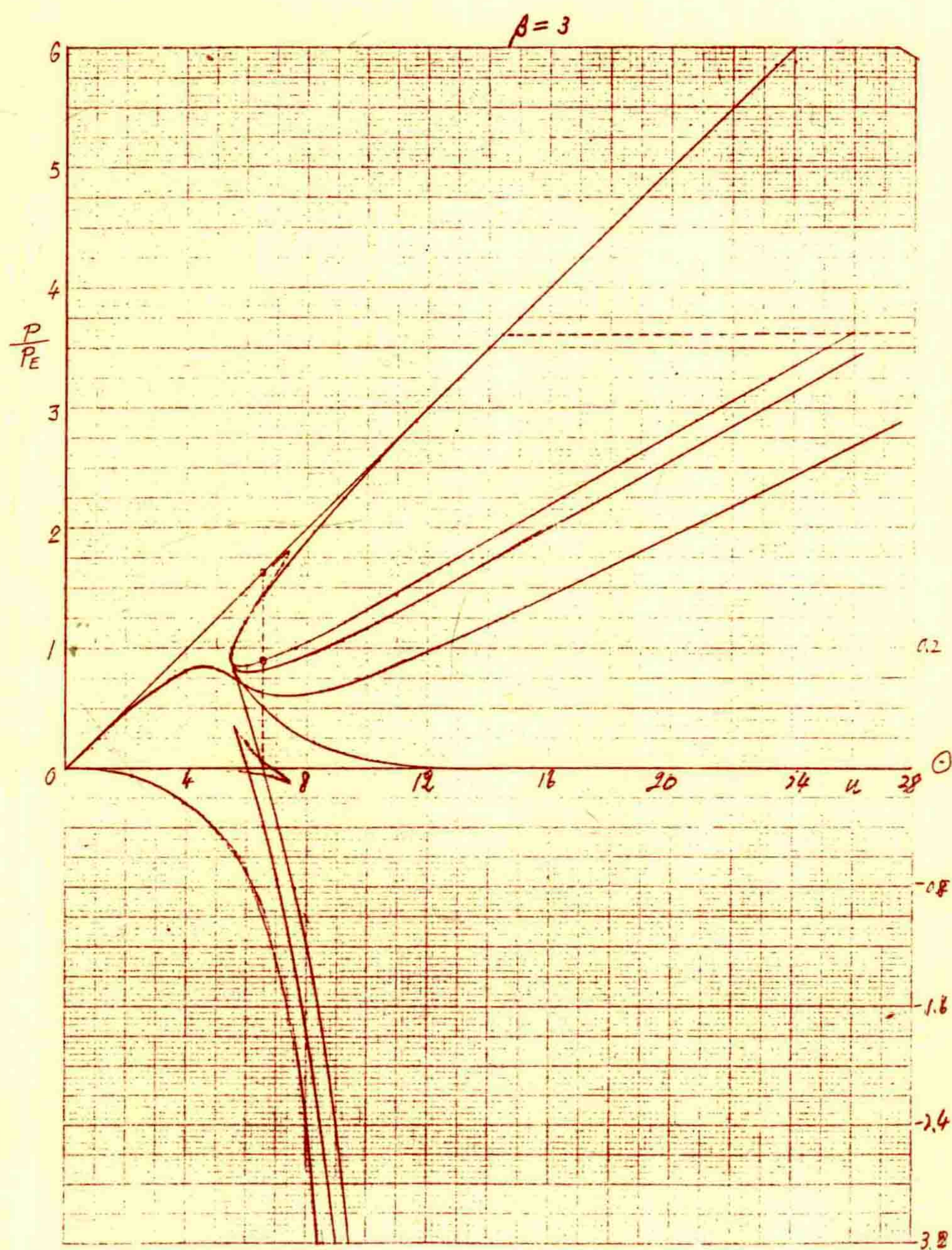
0.2

Θ

0.4

0.6

0.8


β = 3
6
5
4
P/P_E
3
2
1
0
4
8
12
16
20
24
u
28
0.2
-0.8
-1.6
-2.4
-3.2

Section 5

Buckling of Column with One Non-linear Support and Initial Deflection

With Initial Deflection

$$w = \sum_{n=1,3,5}^{\infty} a_n \sin \frac{n\pi x}{l} \; ; \quad w^\circ = a_1^\circ \sin \frac{\pi x}{l}$$

The lowering of the potential of $P = -\frac{Pl}{4}\left[\left(\frac{\pi}{l}\right)^2(a_1^2 - a_1^{\circ 2}) + \sum_{n=3,5,7}^{\infty}\left(\frac{n\pi}{l}\right)^2 a_n^2\right]$

The increase in bending strain energy $= \frac{EI}{2}\int_0^l \left[\frac{d^2w}{dx^2} - \frac{d^2w^\circ}{dx^2}\right]^2 dx$

$$= \frac{EIl}{4}\left[\left(\frac{\pi}{l}\right)^4 (a_1 - a_1^\circ)^2 + \sum_{n=3,5,7}^{\infty}\left(\frac{n\pi}{l}\right)^4 a_n^2\right]$$

$$\delta = (a_1 - a_1^\circ) + \sum_{n=3,5,7}^{\infty} (-)^{\frac{n-1}{2}} a_n$$

Therefore

$$-\frac{Pl}{2}\left(\frac{\pi}{l}\right)^2 a_1 + \frac{EIl}{2}\left(\frac{\pi}{l}\right)^4 (a_1 - a_0^\circ) + F = 0$$

or

$$-\frac{P\pi^2}{2}\frac{a_1}{l} + \frac{P_E\pi^2}{2}\left(\frac{a_1}{l} - \frac{a_1^\circ}{l}\right) + F = 0$$

$$\frac{F}{P_E} = \frac{\pi^2}{2}\left[\left(\frac{P}{P_E} - 1\right)\frac{a_1}{l} + \frac{a_1^\circ}{l}\right]$$

$$\boxed{\frac{a_1}{l} = \frac{\frac{2}{\pi^2}\frac{F}{P_E} - \frac{a_1^\circ}{l}}{\left(\frac{P}{P_E} - 1\right)}}$$

$$\boxed{\frac{a_n}{l} = \frac{2(-)^{\frac{n-1}{2}}}{\pi^2}\,\frac{\frac{F}{P_E}}{n^2\left(\frac{P}{P_E} - n^2\right)}}$$

96

$$\xi = \frac{\frac{2}{\pi^2}\frac{F}{P_E} - \frac{a_1^0}{l}}{\left(\frac{P}{P_E}-1\right)} - \frac{a_1^0}{l} + \frac{2}{\pi^2}\sum_{n=3,5,7}^{\infty}\frac{\frac{F}{P_E}}{n^2\left[\frac{P}{P_E}-n^2\right]}$$

$$\xi = \frac{\frac{2}{\pi^2}\frac{F}{P_E} - \frac{a_1^0}{l}\frac{P}{P_E}}{\left(\frac{P}{P_E}-1\right)} + \frac{2}{\pi^2}\sum_{n=3,5,7}^{\infty}\frac{\frac{F}{P_E}}{n^2\left[\frac{P}{P_E}-n^2\right]}$$

$$\boxed{\xi = \frac{a_1^0}{l}\left(\frac{\frac{P}{P_E}}{1-\frac{P}{P_E}}\right) + \frac{\frac{F}{P_E}}{\frac{P}{P_E}}\left\{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}$$

$$W_1 = \frac{P_E\pi^2 l}{4}\left[\frac{(a_1-a_1^0)^2}{l^2} + \sum_{n=3,5,7}^{\infty} n^4\left(\frac{a_n}{l}\right)^2\right]$$

$$\frac{W_1}{P_E l} = \frac{\pi^2}{4}\left[\left(\frac{\frac{2}{\pi^2}\frac{F}{P_E} - \frac{a_1^0}{l}\frac{P}{P_E}}{\frac{P}{P_E}-1}\right)^2 + \frac{1}{\pi^2}\left(\frac{F}{P_E}\right)^2\sum_{n=3,5,7}^{\infty}\frac{1}{\left(\frac{P}{P_E}-n^2\right)^2}\right]$$

$$\boxed{\frac{W_1}{P_E l} = \frac{\pi^2}{4}\,\frac{\frac{a_1^0}{l}\frac{P}{P_E}\left[\frac{a_1^0}{l}\frac{P}{P_E} - \frac{4}{\pi^2}\frac{F}{P_E}\right]}{\left(\frac{P}{P_E}-1\right)^2} + \frac{1}{\frac{P}{P_E}}\left(\frac{F}{P_E}\right)^2\left[\frac{1}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]}$$

$$\frac{\varepsilon_2}{l} = \frac{\pi^2}{4}\left[1\left\{\left(\frac{a_1}{l}\right)^2 - \left(\frac{a_1^0}{l}\right)^2\right\} + \sum_{n=3,5,7}^{\infty} n^2\left(\frac{a_n}{l}\right)^2\right]$$

$$\boxed{\frac{\varepsilon_2}{l} = \frac{\frac{\pi^2}{4}\left(\frac{a_1^0}{l}\right)^2\left[1-\left(\frac{P}{P_E}-1\right)^2\right] - \frac{F}{P_E}\frac{a_1^0}{l}}{\left(\frac{P}{P_E}-1\right)^2} + \frac{\left(\frac{F}{P_E}\right)^2}{\left(\frac{P}{P_E}\right)^2}\left\{\frac{3}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}}$$

$$\eta = \left(\frac{a_1^0}{\pi i}\right)\frac{\frac{P}{P_E}}{1-\frac{P}{P_E}} + \left\{\frac{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\frac{P}{P_E}}\right\} K\eta\left(1 - 2.08889\eta + 1.04444\eta^2\right)$$

$$\frac{\frac{\varepsilon_2}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{\frac{\pi^2}{4}\left(\frac{a_1^0}{\pi i}\right)^2\left[1-\left(\frac{P}{P_E}-1\right)^2\right] - \left(\frac{\frac{F}{P_E}}{\pi\frac{i}{l}}\right)\left(\frac{a_1^0}{\pi i}\right)}{\left(\frac{P}{P_E}-1\right)^2} + \frac{\left(\frac{\frac{F}{P_E}}{\pi\frac{i}{l}}\right)^2}{\left(\frac{P}{P_E}\right)^2}\left\{\frac{3}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}$$

$$\frac{\frac{W_1}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{\pi^2}{4}\,\frac{\left(\frac{a_1^0}{\pi i}\right)\frac{P}{P_E}\left[\left(\frac{a_1^0}{\pi i}\right)\frac{P}{P_E} - \frac{4}{\pi^2}\frac{\frac{F}{P_E}}{\frac{\pi i}{l}}\right]}{\left(\frac{P}{P_E}-1\right)^2} + \left(\frac{\frac{F}{P_E}}{\frac{\pi i}{l}}\right)^2\frac{\left[\frac{1}{16} + \frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{8\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]}{\frac{P}{P_E}}$$

Put $\frac{a_1^0}{\pi i} = 0.5$

$$\frac{\frac{\varepsilon_1}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \left(\frac{P}{P_E}\right)$$

$$\frac{\frac{W_2}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = K\eta^2\left[\frac{1}{2} - 0.6962962962\eta + 0.26111111\eta^2\right]$$

$$\frac{\frac{W_3}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{1}{2}\left(\frac{P}{P_E}\right)^2$$

①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪
$\sqrt{\frac{P}{P_E}}$	P/P_E	$\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$	$\frac{1}{4}-\frac{1}{2\pi\sqrt{P_E}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$	④$/(P/P_E)$	$(P/P_E)/1-(\frac{P}{P_E})$	$\frac{3}{4}$④$+\frac{1}{16}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$	$(P/P_E)^2$	⑦/⑧	⑦$-\frac{1}{2}$④	⑩$/(P/P_E)$
1.9	3.61	−0.15838	+0.263267	0.072927	−1.38314	0.199018	13.0321	0.015271	0.067385	0.018666
1.8	3.24	−0.32492	+0.278729	0.086027	−1.44643	0.215645	10.4976	0.020542	0.076281	0.023544
1.7	2.89	−0.50953	+0.297702	0.103011	−1.52910	0.239503	8.3521	0.028676	0.090652	0.031367
1.6	2.56	−0.72654	+0.322270	0.125887	−1.64103	0.274694	6.5536	0.041915	0.113559	0.044359
1.5	2.25	−1.0000	+0.356103	0.158268	−1.80000	0.329577	5.0625	0.065102	0.151526	0.067345
1.4	1.96	−1.3764	+0.406472	0.207384	−2.04167	0.423259	3.8416	0.110178	0.220023	0.112257
1.3	1.69	−1.9626	+0.490275	0.290104	−2.44928	0.608444	2.8561	0.213033	0.363307	0.214975
1.2	1.44	−3.0777	+0.658193	0.457078	−3.27273	1.085660	2.0736	0.523563	0.756564	0.525392
1.1	1.21	−6.3138	+1.163521	0.961587	−5.76190	3.366145	1.4641	2.297756	2.782385	2.299492
1.0	1.00	−∞ (+∞)	+∞ (−∞)	∞	−∞	∞	1.0000	∞	∞	∞
0.9	0.81	+6.3138	−0.866525	−1.069784	+4.26316	1.841610	0.6561	2.806904	2.274873	2.808485
0.8	0.64	+3.0777	−0.362289	−0.566077	+1.77778	0.320298	0.4096	0.781978	0.501443	0.783505
0.7	0.49	+1.9626	−0.196224	−0.400457	+0.96078	0.093569	0.2401	0.389708	0.191681	0.391186
0.6	0.36	+1.3764	−0.115101	−0.319725	+0.56250	0.032079	0.1296	0.247523	0.089630	0.248972
0.5	0.25	+1.0000	−0.068310	−0.273240	+0.33333	0.011268	0.0625	0.180288	0.045423	0.181692
0.4	0.16	+0.72654	−0.039081	−0.244256	+0.19048	0.0036803	0.0256	0.143762	0.023221	0.145131

98

Calculations of η $\qquad K = 13.7123$

$\frac{P}{P_E} = 0.16;$

$$\eta = 0.09524 - 3.34931\,\eta\,(1 - 2.08889\,\eta + 1.04444\,\eta^2)$$

$$3.49815\,\eta^3 - 6.99633\,\eta^2 + 4.34931\,\eta - 0.09524 = 0$$

$$\eta^3 - 2.0000\,\eta^2 + 1.24332\,\eta - 0.02723 = 0.$$

$$F'(\eta) = 3\eta^2 - 4.000\,\eta + 1.24332$$

$F(0.0228) = +0.00009 \qquad F'(0.0228) = 1.15368$

$F(0.02272) = $ O.K. $\qquad \underline{\eta = 0.02272}$

$\eta^2 - 1.97728\,\eta + 1.19840 = 0, \qquad$ No real root!

$$\frac{\left(\frac{F}{P_E}\right)}{\left(\pi\frac{i}{l}\right)} = K\eta\,(1 - 2.08889\,\eta + 1.04444\,\eta^2)$$

$$= 0.29693$$

$\Theta = \underline{-0.00638}$

$$\frac{\frac{\varepsilon_2}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{0.61685 \times 0.2944 - 0.14847}{0.7056} + 0.29693^2 \times 0.143762$$

$$= 0.04695 + 0.01268 = \underline{0.05963}$$

$\frac{\frac{\varepsilon_{TOT}}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{0.21963}; \qquad \frac{\frac{M_{TOT}}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{0.01774}$

$$\frac{\frac{M_1}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{2.46740}{0.7056}\left[0.08\left(0.08 - \frac{0.29693}{2.46740}\right)\right] + 0.29693^2 \times 0.145131$$

$= -0.011285 + 0.012796 = \underline{+0.001511}, \quad \frac{\frac{M_2}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{+0.003428}, \quad \frac{\frac{M_3}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{0.01280}$

$\frac{P}{P_E} = \underline{0.25}$

$\eta = 0.1666667 - 3.74675\left(1 - 2.08889\eta + 1.04444\eta^2\right)\eta$

$3.91327\eta^3 - 7.82654\eta^2 + 4.74675\eta - 0.166667 = 0$

$\eta^3 - 2\eta^2 + 1.21299\eta - 0.04259 = 0$

$F'(\eta) = 3\eta^2 - 4\eta + 1.21299 \; ; \qquad F(0.0373) = -0.00008 \; ; \; F'(0.0373) = 1.0679$

$\underline{\eta = 0.03738}$

$\eta^2 - 1.96262\eta + 1.13963 = 0$ No Real root

$\frac{\left(\frac{F}{P_E}\right)}{\left(\pi\frac{i}{l}\right)} = \underline{0.47329}$, $\qquad \frac{\frac{\varepsilon_2}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{0.61685 \times 0.4375 - 0.23665}{0.5625} + 0.47329^2 \times 0.180248$

$= 0.05906 + 0.04038 = \underline{0.09944} \qquad \frac{\frac{\varepsilon_{TOT}}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{0.34944}$

$\frac{\frac{W_1}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{2.46740}{0.5625}\left[0.125\left(0.125 - \frac{0.47329}{2.46740}\right)\right] + 0.47329^2 \times 0.181692$

$= -0.036639 + 0.040699 = \underline{0.004060} \qquad \frac{\frac{W_2}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{+0.009088} \qquad \frac{\frac{W_3}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{0.03125}$

$\frac{\frac{W_{TOT}}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{0.044398} \qquad \Theta = \underline{-0.016656}$

$\frac{P}{P_E} = 0.36$

$$\eta = 0.28125 - 4.38417\,(1 - 2.088889\,\eta + 1.04444\,\eta^2)\,\eta$$

$$4.57900\,\eta^3 - 9.15804\,\eta^2 + 5.38417\,\eta - 0.28125 = 0$$

$$\eta^3 - 2\eta^2 + 1.17584\,\eta - 0.06142 = 0$$

$$F'(\eta) = 3\eta^2 - 4\eta + 1.17584 \;; \quad F(0.058) = +0.00025 \;; \quad F'(0.058) = 0.954$$

$$\eta = \underline{0.05774} \qquad \eta^2 - 1.94226\,\eta + 1.06369 = 0$$

$$\frac{\frac{E}{P_E}}{\pi \frac{i}{l}} = \underline{0.69901} \;; \quad \frac{\frac{\varepsilon_2}{l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \frac{0.61685 \times 0.5904 - 0.34951}{0.4096} + 0.69901^2 \times 0.247523$$

$$= 0.03584 + 0.12094 = \underline{0.15678}$$

$$\frac{\frac{\varepsilon_{TOT}}{l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \underline{0.51678}$$

$$\frac{\frac{W_1}{P_E l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \frac{2.46740}{0.4096}\left[0.18\left(0.18 - \frac{0.69901}{2.46740}\right)\right] + 0.69901^2 \times 0.248972 = -0.11201 + 0.12165$$

$$= \underline{0.00964}$$

$$\frac{\frac{W_2}{P_E l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \underline{0.08406} \;; \quad \frac{\frac{W_{TOT}}{P_E l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \underline{0.09550}$$

$$\Theta = \underline{-0.03803}$$

$$\frac{P}{P_E} = 0.49$$

$$\eta = 0.48039 - 5.49119\,(1 - 2.08889\eta + 1.04444\eta^2)\eta$$

$$5.73522\eta^3 - 11.47048\eta^2 + 6.49119\eta - 0.48039 = 0$$

$$\eta^3 - 2\eta^2 + 1.13181\eta - 0.08376 = 0$$

$$F'(\eta) = 3\eta^2 - 4\eta + 1.13181\ ; \qquad F(0.086) = -0.0058\ ; \quad F'(0.086) = 0.81$$

$$\underline{\eta = 0.08671}\ ; \qquad \eta^2 - 1.91329\eta + 0.96591 = 0\ ; \quad \underline{\text{no real root}}$$

$$\left(\frac{\frac{E}{P_E}}{\pi\frac{i}{l}}\right) = \underline{0.98296}\ ; \qquad \frac{\frac{\varepsilon_2}{l}}{\pi^2(\frac{i}{l})^2} = \frac{0.61685 \times 0.7399 - 0.49148}{0.2601} + 0.98296^2 \times 0.389708$$

$$= -0.13484 + 0.37654 = \underline{0.24170} \qquad \frac{\frac{\varepsilon_{TOT}}{l}}{\pi^2(\frac{i}{l})^2} = \underline{0.73170}$$

$$\frac{\frac{W_1}{P_E l}}{\pi^2(\frac{i}{l})^2} = \frac{2.46740}{0.2601}\left\{0.245\left(0.245 - \frac{0.98296}{2.46740}\right)\right\} + 0.98296^2 \times 0.391186$$

$$= -0.35648 + 0.37797 = \underline{0.02149}\ ; \qquad \frac{\frac{W_2}{P_E l}}{\pi^2(\frac{i}{l})^2} = \underline{0.04553}$$

$$\frac{\frac{W_{TOT}}{P_E l}}{\pi^2(\frac{i}{l})^2} = \underline{0.18707} \qquad \Theta = -\underline{0.080622}$$

$\frac{P}{P_E} = 0.64$

$$\eta = 0.888889 - 7.76222(1 - 2.011119\eta + 1.04444\eta^2)\eta$$

$$8.10720\eta^3 - 16.21442\eta^2 + 8.76222\eta - 0.88889 = 0$$

$$\eta^3 - 2\eta^2 + 1.08079\eta - 0.10964 = 0$$

$$F'(\eta) = 3\eta^2 - 4\eta + 1.08079 \;; \quad F(0.131) = -0.00013; \quad F'(0.131) = 0.608$$

$$\underline{\eta_1 = 0.13121} \qquad \eta^2 - 1.86879\eta + 0.83559 = 0 \left\{ \begin{array}{l} \eta_2 = 0.93440 - 0.19369 \\ \eta_3 = 0.93440 + 0.19369 \end{array} \right.$$

$$\underline{\eta_2 = 0.74071}$$

$$\underline{\eta_3 = 1.12809}$$

$$\left(\frac{\frac{F}{P_E}}{\pi\frac{i}{\ell}}\right) = \underline{1.33842}, \quad \frac{\frac{\varepsilon_2}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{0.61685 \times 0.8704 - 0.66921}{0.1296} + 1.33842^2 \times 0.781978$$

$$= -1.02086 + 1.40081 = \underline{0.37995} \qquad \frac{\frac{\varepsilon_{TOT}}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \underline{1.0200}$$

$$\frac{\frac{W_1}{P_E\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{2.46740}{0.1296}\left[0.32\left(0.32 - \frac{1.33842}{2.46740}\right)\right] + 1.33842^2 \times 0.783505 = -1.35518 + 1.40355$$

$$= \underline{0.04837}$$

$$\frac{\frac{W_2}{P_E\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \underline{0.097528} \qquad \frac{\frac{W_{TOT}}{P_E\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \underline{0.35070} \qquad \Theta = -\underline{0.1695}$$

$$\frac{\frac{F}{P_E}}{\pi\left(\frac{i}{\ell}\right)} = \underline{0.26177}, \quad \frac{\frac{\varepsilon_2}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{0.61685 \times 0.8704 - 0.13089}{0.1296} + 0.26177^2 \times 0.781978$$

$$= 3.13287 + 0.05358 = \underline{3.18645}, \quad \frac{\frac{\varepsilon_{TOT}}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \underline{3.82685}$$

$$\frac{\frac{W_1}{P_E\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{2.46740}{0.1296}\left[0.32\left(0.32 - \frac{0.26177}{2.46740}\right)\right] + 0.26177^2 \times 0.783505 = 1.30321 + 0.05369 = \underline{1.35690}$$

$$\frac{\frac{W_2}{P_E\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \underline{0.95926} \qquad \frac{\frac{W_{TOT}}{P_E\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \underline{2.52096}; \qquad \Theta = \underline{-4.7999}$$

$$\frac{\frac{F}{P_E}}{\pi\left(\frac{i}{l}\right)} = \underline{-0.42261} \qquad \frac{\frac{\varepsilon_2}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{0.61685 \times 0.8704 + 0.21131}{0.1296} + 0.42261^2 \times 0.781978$$

$$= 5.77327 + 0.13966 = \underline{5.91293} \qquad \frac{\frac{\varepsilon_{TOT}}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{6.55293}$$

$$\frac{P}{P_E} = \underline{0.81} \qquad \eta = 2.13158 - 14.66920\left(1 - 2.08889\,\eta + 1.04444\,\eta^2\right)\eta$$

$$15.32116\,\eta^3 - 30.64233\,\eta^2 + 15.66920\,\eta - 2.13158 = 0$$

$$\eta^3 - 2\eta^2 + 1.02271\,\eta - 0.13913 = 0$$

$$F'(\eta) = 3\eta^2 - 4\eta + 1.02271\,; \qquad F(0.22) = -0.00029\,; \quad F'(0.22) = 0.288$$

$$F(0.221) = \text{O.K.}$$

$$\eta_1 = \underline{0.22100}\,; \qquad \eta^2 - 1.77900\,\eta + 0.62955 = 0. \qquad \eta = 0.88950 \pm \sqrt{0.16166}$$

$$\eta_2 = \underline{0.48743}\,; \qquad\qquad = 0.88950 \pm 0.40207$$

$$\eta_3 = \underline{1.29157}\,;$$

$$\left(\frac{\frac{F}{P_E}}{\pi\frac{i}{l}}\right) = \underline{1.78604}\,; \quad \frac{\frac{\varepsilon_1}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{0.61685 \times 0.9639 - 0.89302}{0.0361} + 1.78604^2 \times 2.806904$$

$$= -8.26199 + 8.95386 = \underline{0.68687} \qquad \frac{\frac{\varepsilon_{TOT}}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{1.49687}$$

$$\frac{\frac{W_1}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{2.46740}{0.0361}\left[0.405\left(0.405 - \frac{1.78604}{2.46740}\right)\right] + 1.78604^2 \times 2.808485 = -8.82648 + 8.95890$$

$$= \underline{0.13242}$$

$$\frac{\frac{W_2}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{0.240364} \qquad \frac{\frac{W_{TOT}}{P_E l}}{\pi^2\left(\frac{i}{l}\right)^2} = \underline{0.70081} \qquad \Theta = -\underline{0.42087}$$

$$\left(\frac{\frac{F}{P_E}}{\pi\frac{i}{l}}\right) = \underline{1.53701} \qquad \frac{\frac{\varepsilon_1}{l}}{(\pi)^2\left(\frac{i}{l}\right)^2} = \frac{0.61685 \times 0.9639 - 0.76851}{0.0361} + 1.53701^2 \times 2.806904 = \begin{array}{r} -4.81796 \\ +6.63103 \\ \hline +1.81307 \\ 0.81 \\ \hline 2.62307 \end{array}$$

$$\frac{\frac{F}{P_E}}{\pi\left(\frac{i}{\ell}\right)} = \underline{0.78527}, \quad \frac{\frac{\varepsilon_1}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{0.61685 \times 0.9639 - 0.392635}{0.0361} + 0.78527^2 \times 2.806904$$

$$= 5.59395 + 1.73088 = 7.3248, \quad \frac{\frac{\varepsilon_{TOT}}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \underline{8.1348}$$

---

when $\sqrt{\frac{P}{P_E}} = 1 - \varepsilon$

$$\frac{\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}(1-\varepsilon)}{P/P_E} = -\frac{1}{2\pi}\frac{\cos\frac{\varepsilon}{2}\pi}{\sin\frac{\varepsilon}{2}\pi} = -\frac{1}{2\pi}\frac{1 - \frac{1}{2!}\left(\frac{\varepsilon}{2}\right)^2\pi^2 + \cdots}{\frac{\varepsilon\pi}{2}\left(1 - \frac{1}{3!}\left(\frac{\varepsilon}{2}\right)^2\pi^2 + \cdots\right)}$$

$$= -\frac{1}{\varepsilon\pi^2}$$

$$\frac{P/P_E}{1 - P/P_E} = \frac{1}{1-(1-\varepsilon)^2} = \frac{1}{1-(1-2\varepsilon+\varepsilon^2)} = \frac{1}{2\varepsilon}\frac{1}{\left(1-\frac{\varepsilon}{2}\right)}$$

the equation for η.

$$0 = 0.25000 - 1.38935\left(1 - 2.08889\eta + 1.04444\eta^2\right)\eta$$

$$1.04444\eta^3 - 2.08889\eta + \eta - 0.17994 = 0$$

$$\eta^3 - 2\eta^2 + 0.95745\eta - 0.17228 = 0$$

$$F'(\eta) = 3\eta^2 - 4\eta + 0.95745; \qquad F(1.4061) = -0.00021; \quad F'(1.4061) = 1.264$$

$\eta = \underline{1.40627}$, $\quad \eta^2 - 0.59373\eta + 0.12251 = 0$ only one real root!

with $\frac{a_1^0}{\pi i} = 0.5$, $\beta = 3$

$P/P_E = 0.16$

$u = 0.17491$ $\qquad \frac{\frac{W}{P_E l}}{(\pi^2 \frac{i}{l})^2 + \alpha} = 0.014035$ $\qquad \theta = -0.001262$

$P/P_E = 0.25$ $\qquad u = 0.27466$ $\qquad \theta = -0.003237$

$P/P_E = 0.36$ $\qquad u = 0.39920$ $\qquad \theta = -0.007205$

$P/P_E = 0.49$ $\qquad u = 0.55043$ $\qquad \theta = -0.014681$

$P/P_E = 0.64$ $\qquad u = 0.73499$ $\qquad \theta = -0.028830$ $\qquad u = 1.6066$ $\quad \theta = 0.21572$

$u = 2.1182$

$P/P_E = 0.81$ $\qquad u = 0.98172$ $\qquad \theta = -0.06046$

$u = 1.26327$ $\qquad \theta =$

$u = 2.6412$

$$\frac{\frac{W_1}{P_E l}}{\pi^2 (\frac{i}{l})} = \frac{2.46740}{0.1296}\left[0.32\left(0.32 + \frac{0.42261}{2.46740}\right)\right] + 0.42261^2 \times 0.783505$$

$$= 2.99306 + 0.13993 = 3.13299$$

$$\frac{\frac{W_2}{P_E l}}{\pi^2 (\frac{i}{l})^2} = 1.60215$$

$\beta = 3$ $\qquad \theta = 0.2548$

$\frac{P}{P_E} = \underline{1.21}$ $\eta = -2.88095 + 13.18557(1 - 2.08889\eta + 1.04444\eta^2)\eta$

$13.77159\eta^3 - 27.54319\eta^2 + 12.18557\eta - 2.88095 = 0.$

$\eta^3 - 2\eta^2 + 0.88483\eta - 0.20919 = 0.$

$F'(\eta) = 3\eta^2 - 4\eta + 0.88483$; $F(1.5042) = -0.00003$; $F'(\eta) = 1.82$

$\underline{\eta = 1.50422}$ $\left(\frac{\frac{E}{P_E}}{\pi i}\right) = \underline{4.56048}$

$$\frac{\frac{\varepsilon_2}{l}}{\pi\left(\frac{i}{l}\right)^2} = \frac{0.61685 \times 0.9559 - 2.28024}{0.0441} + 4.56048^2 \times 2.297756$$

$$= -38.3354 + 47.78868 = \underline{9.45324}$$

for $\beta = 3$; $u = \underline{3.57331}$

$P/P_E = 2.25$; $\eta = -0.90000 + 2.17022(1 - 2.08889\eta + 1.04444\eta^2)\eta$

$2.26667\eta^3 - 4.43689\eta^2 + 1.17022\eta - 0.9000 = 0$

$\eta^3 - 2\eta^2 + 0.51627\eta - 0.397058 = 0$

$F'(\eta) = 3\eta^2 - 4\eta + 0.51627$; $F(1.8365) = -0.00038$; $F'(1.8365) = 3.49$

$\eta = \underline{1.83667}$ $\left(\frac{\frac{E}{P_E}}{\pi i}\right) = \underline{17.2904}$

$$\frac{\frac{\varepsilon_2}{l}}{\pi\left(\frac{i}{l}\right)^2} = \frac{-0.61685 \times 0.5625 - 8.6452}{1.5625} + 17.2904^2 \times 0.065102$$

$$= -5.75499 + 19.46276 = 13.70777$$

for $\beta = 3$; $u = \underline{5.6769}$

$$\boxed{\frac{a_1^0}{\pi i} = 0.10}$$

$P/P_E = \underline{1.96}$ $\qquad \eta = -0.20417 + 2.84371(1 - 2.08889\eta + 1.04444\eta^2)\eta$

$$2.97010\eta^3 - 5.94020\eta^2 + 1.84371\eta - 0.20417 = 0$$

$$\eta^3 - 2\eta^2 + 0.62076\eta - 0.068742 = 0$$

$F'(\eta) = 3\eta^2 - 4\eta + 0.62076$; $\qquad F(1.6486) = 0.000424$; $F'(1.6486) = 2.18$

$\eta = \underline{1.64879}$ $\qquad \eta^2 - 0.35121\eta + 0.04169 = 0$ no real roots

$$\frac{\frac{\varepsilon_2}{l}}{\pi^2\left(\frac{i}{l}\right)^2} = \frac{0.024674 \times 0.0784 - 0.893428}{0.9216} + 8.93428^2 \times 0.110178 = -0.96733 + 8.79456 = \underline{7.82723}$$

for $\beta = 0$; $\quad n = \underline{9.78723}$

$$\left(\frac{\frac{F}{P_E}}{\pi\frac{i}{l}}\right) = 8.93428 \qquad \beta = 3 ; \quad n = \underline{3.9168}$$

$P/P_E = \underline{1.69}$ $\qquad \eta = -0.24493 - 3.97799(1 - 2.08889\eta + 1.04444\eta^2)\eta$

$$4.15479\eta^3 - 8.30958\eta^2 + 2.97799\eta - 0.24493 = 0$$

$$\eta^3 - 2\eta^2 + 0.71677\eta - 0.058952 = 0$$

$F(\eta) = 3\eta^2 - 4\eta + 0.71677$, $\qquad F(1.566) = -0.00080$; $F'(1.566) = 1.80$

$\eta_1 = \underline{1.56645}$; $\qquad \eta^2 - 0.43355\eta + 0.03764 = 0$

$\eta_2 = \underline{0.12007}$ $\qquad \eta = 0.21678 \pm \sqrt{0.009353} = 0.21678 \pm 0.09671$

$\eta_3 = \underline{0.31349}$

$\eta = 0.12007$

$\left(\frac{\frac{F}{P_E}}{\pi \frac{i}{l}}\right) = 1.25827$; $\frac{\frac{\varepsilon_2}{l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \frac{0.024674 \times 0.5239 - 0.125827}{0.4761} + 1.25827^2 \times 0.213033$

$= -0.23714 + 0.33728 = 0.10014$;

for $\beta = 0$; $u = 1.79014$

$\beta = 3$; $u = 1.71504$

$\frac{\frac{W_1}{P_E l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \frac{2.46740}{0.4761}\left[0.169\left(0.169 - \frac{1.25827}{2.4674}\right)\right] + 1.25827^2 \times 0.214975$

$= -0.29863 + 0.34036 = \underline{0.04173}$;

$\frac{\frac{W_2}{P_E l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \underline{0.08306}$; $\beta = 0$ $\Theta = -\underline{0.04946}$

$\beta = 3$; $\Theta = -\underline{0.01143}$

$\eta = \underline{0.31349}$

$\left(\frac{\frac{F}{P_E}}{\pi \frac{i}{l}}\right) = 1.92495$; $\frac{\frac{\varepsilon_2}{l}}{\pi \left(\frac{i}{l}\right)^2} = \frac{0.024674 \times 0.5239 - 0.192495}{0.4761} + 1.92495^2 \times 0.213033$

$= -0.37718 + 0.78938 = 0.41220$; $\beta = 0$ $u = 2.10220$; $\beta = 3$ $u = 1.79305$

$\frac{\frac{W_1}{P_E l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \frac{2.46740}{0.4761}\left[0.169\left(0.169 - \frac{1.92495}{2.46740}\right)\right] + 1.92495^2 \times 0.214975 = -0.53527 + 0.79657$

$= \underline{0.26130}$

$\frac{\frac{W_2}{P_E l}}{\pi^2 \left(\frac{i}{l}\right)^2} = \underline{0.41422}$ $\beta = 0$ $\Theta = -\underline{0.10605}$

$\beta = 3$ $\Theta = -\underline{0.01059}$

$\eta = 1.56645$

$$\left(\frac{\frac{F}{P_E}}{i\frac{\pi}{\ell}}\right) = 6.24327 \quad \frac{\frac{\varepsilon_2}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{0.024674 \times 0.5239 - 0.624327}{0.4761} + 6.24327^2 \times 0.213033$$

$$= -1.28419 + 8.30369 = 7.01950, \quad \beta = 3 \quad u = 3.4487$$

$P/P_E = 1.44$ $\qquad \eta = -0.327273 + 6.26759\,(1 - 2.08889\eta + 1.04444\eta^2)\eta$

$$6.54615\eta^3 - 13.09230\eta^2 + 5.26759\eta - 0.327273 = 0$$

$$\eta^3 - 2\eta^2 + 0.80469\eta - 0.04999 = 0$$

$F'(\eta) = 3\eta^2 - 4\eta + 0.80469$; $\quad F(0.075) = -0.00047 \quad F'(0.075) = 0.5216$

$\eta_1 = 0.07590$; $\qquad \eta^2 - 1.92410\eta + 0.65865 = 0$

$\eta_2 = 0.44544 \qquad \eta = 0.96205 \pm \sqrt{0.26689} = 0.96205 \pm 0.51661$

$\eta_3 = 1.47866$

$\eta = 0.07590$

$$\left(\frac{\frac{F}{P_E}}{\pi\frac{i}{\ell}}\right) = 0.88202; \quad \frac{\frac{\varepsilon_2}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{0.024674 \times 0.8064 - 0.088202}{0.1936} + 0.88202^2 \times 0.523563$$

$= -0.352404 + 0.40731 = +0.05451 \qquad \beta = 0 \quad u = 1.49451$

$\qquad \beta = 3 \quad u = 1.45363$

$$\frac{\frac{W_1}{P_E \ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{2.46740}{0.1936}\left[0.144\left(0.144 - \frac{0.88202}{2.46740}\right)\right] + 0.88202^2 \times 0.525392 = -0.39177 + 0.40873$$

$$= 0.01696$$

$$\frac{\frac{W_2}{P_E \ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = 0.03544; \qquad \beta = 0 \quad \Theta = -0.02757$$

$\qquad \beta = 3 \quad \Theta = -0.00662$

$\eta = 0.44544$

$$\left(\frac{\frac{F}{P_E}}{\pi^2 \frac{i}{l}}\right) = 1.69044 \qquad \frac{\frac{\varepsilon_2}{l}}{\pi^2 (\frac{i}{l})^2} = \frac{0.024674 \times 0.1064 - 0.169044}{0.1936} + 1.69064 \times^2 0.523563$$

$$= -0.77037 + 1.49613 = 0.72576 \qquad \beta = 0 \quad u = 2.16576$$

$$\beta = 3 \quad u = 1.62144$$

$$\frac{\frac{V_2}{P_E l}}{\pi^2 (\frac{i}{l})^2} = \frac{2.46740}{0.1936}\left[0.144\left(0.144 - \frac{1.69044}{2.4674}\right)\right] + 1.69044^2 \times 0.525392 = -0.99308 + 1.50135$$

$$= 0.50827$$

$$\frac{\frac{W_2}{P_E l}}{\pi^2 (\frac{i}{l})^2} = 0.65747 \qquad \beta = 0 \quad \Theta = -0.14272$$

$$\beta = 3 \quad \Theta = +0.01371$$

$\eta = 1.47866$

$$\frac{\frac{F}{P_E}}{\pi^2 (\frac{i}{l})^2} = 3.95054 \qquad \frac{\frac{\varepsilon_2}{l}}{\pi^2 (\frac{i}{l})} = \frac{0.024674 \times 0.1064 - 0.395054}{0.1936} + 3.95054^2 \times 0.523563$$

$$= -1.93777 + 8.17113 = 6.23336 \; ; \quad \beta = 0 \quad u = 7.67336$$

$$\beta = 3 \quad u = 2.99834$$

$\boxed{P/P_E = 1.21}$

$$\eta = -0.57619 + 13.18557\,(1 - 2.08819\,\eta + 1.04444\,\eta^2)\,\eta$$

$$13.77159\,\eta^3 - 27.54319\,\eta^2 + 12.18557 - 0.57619 = 0$$

$$\eta^3 - 2\eta^2 + 0.88483\,\eta + 0.04184 = 0$$

$$F'(\eta) = 3\eta^2 - 4\eta + 0.88483 \; ; \qquad F(0.053) = -0.00041 \; ; \quad F'(0.053) = 0.681$$

$$\eta_1 = 0.05360 \qquad \eta^2 - 1.94640\,\eta + 0.78050 = 0 \qquad \eta = 0.97320 \pm \sqrt{0.16662}$$

$$= 0.97320 \pm 0.40822$$

$$\eta_2 = 0.56498$$

$$\eta_3 = 1.38142$$

$\eta = 0.56498$; $\dfrac{\left(\frac{F}{P_E}\right)}{i\pi/\ell} = 1.18694$

$$\frac{\frac{\varepsilon_2}{\ell}}{\pi\left(\frac{i}{\ell}\right)^2} = \frac{0.024674 \times 0.9559 - 0.118694}{0.0441} + 1.18694^2 \times 2.297756$$

$$= -2.15665 + 3.23715 = 1.08050$$

$\beta = 0$ $u = 2.29050$

$\beta = 3$ $u = 1.48012$

$$\frac{\frac{W_1}{P_E \ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{2.46740}{0.0441}\left[0.121\left(0.121 - \frac{1.18694}{2.46740}\right)\right] + 1.18694^2 \times 2.299492$$

$$= -2.43752 + 3.23959 = \underline{0.80207}$$

$$\frac{\frac{W_2}{P_E \ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \underline{0.83143}$$

$\beta = 0$ $\Theta = -0.25765$

$\beta = 3$ $\Theta = +0.04505$

$\eta = 1.38142$ $\dfrac{\frac{F}{P_E}}{\pi\frac{i}{\ell}} = 2.03613$

$$\frac{\frac{\varepsilon_2}{\ell}}{\pi^2\left(\frac{i}{\ell}\right)^2} = \frac{0.024674 \times 0.9559 - 0.203613}{0.0441} + 2.03613^2 \times 2.297756 = -4.08225 + 9.52611$$
$$= 5.44386$$

$\beta = 0$ $u = 6.65386$

$\beta = 3$ $u = 2.57096$

$\underline{P/P_E = 0.81}$ $\eta = 40.426316 - 14.6691(1 - 2.088889\eta + 1.044444\eta^2)\eta$

$$15.3211\,\eta^3 - 30.6421\,\eta^2 + 15.6691\,\eta - 0.426316 = 0$$

$\eta^3 - 2\eta^2 + 1.022717\eta - 0.02783 = 0$;

$F'(\eta) = 3\eta^2 - 4\eta + 1.02271$; $F(0.029) = +0.00017$; $F'(0.029) = 0.909$

$\eta_1 = \underline{0.02881}$; $\eta^2 - 1.97119\eta + 0.96592 = 0$; $\eta = 0.98560 \pm \sqrt{0.00548}$

$\eta_2 = \underline{0.91152}$ $= 0.98560 \pm 0.07408$

$\eta_3 = \underline{1.05968}$

$\eta = 0.91152$ $\left(\frac{\frac{E}{P_E}}{\pi \frac{i}{l}}\right) = -0.45330$

$$\frac{\frac{E_2}{l}}{\pi^2 (\frac{i}{l})^2} = \frac{0.024674 \times 0.9139 + 0.045330}{0.0361} + 0.45330^2 \times 2.806904 = 1.91449 + 0.57676 = 2.49125$$

$\beta = 0$ $u = 3.30125$

$\beta = 3$ $u = 1.43281$

$$\frac{\frac{W_1}{P_E l}}{\pi^2 (\frac{i}{l})^2} = \frac{2.4674}{0.0361}\left[0.081\left(0.081 + \frac{0.45330}{2.46740}\right)\right] + 0.45330^2 \times 2.808485$$

$$= 1.46556 + 0.57709 = 2.04265$$

$\frac{\frac{W_2}{P_E l}}{\pi^2 (\frac{i}{l})^2} = 0.93722$ $\beta = 0$ $\Theta = -2.14121$

$\beta = 3$ $\Theta = +0.04655$

$\eta = 1.05968$; $\left(\frac{\frac{E}{P_E}}{\pi \frac{i}{l}}\right) = -0.59183$

$$\frac{\frac{E_2}{l}}{\pi^2 (\frac{i}{l})^2} = \frac{0.024674 \times 0.9139 + 0.059183}{0.0361} + 0.59183^2 \times 2.806904 = 2.29823 + 0.98315 = 3.28138$$

$\beta = 0$ $u = 4.09138$

$\beta = 0$ $u = 1.63034$

$$\frac{\frac{W_1}{P_E l}}{\pi^2 (\frac{i}{l})^2} = \frac{2.4674}{0.0361}\left[0.081\left(0.081 + \frac{0.59183}{2.4674}\right)\right] + 0.59183^2 \times 2.808485$$

$= 1.77638 + 0.98370 = 2.76008$ $\frac{\frac{W_2}{P_E l}}{\pi^2 (\frac{i}{l})^2} = 0.85236$

$\beta = 0$ $u = -$

$\beta = 3$ $u = -0.07784$

Second Variation of Potential !!!

we have
$$\frac{\partial W}{\partial a_n} = \frac{1}{2}\left(\frac{\pi}{l}\right)^2 P_E\, n^2\left[n^2 - \frac{P}{P_E}\right] a_n + \sin\frac{n\pi}{2}\, F$$

Therefore
$$\frac{1}{P_E l}\frac{\partial^2 W}{\partial a_n^2} = \frac{1}{2}\left(\frac{\pi}{l}\right)^2 n^2\left[n^2 - \frac{P}{P_E}\right] + \frac{1}{l}\left(\sin\frac{n\pi}{2}\right)\frac{d\frac{F}{P_E}}{d(\delta/l)}\frac{\partial(\delta/l)}{\partial a_n}$$

Put
$$\frac{a_n}{l} = b_n$$

$$\frac{1}{P_E l}\frac{\partial^2 W}{\partial b_n^2} = \frac{\pi^2}{2} n^2\left[n^2 - \frac{P}{P_E}\right] + \left(\sin^2\frac{n\pi}{2}\right)\frac{F'}{P_E}\left(\frac{\delta}{l}\right)$$

where
$$\frac{F'}{P_E} = \frac{d(F/P_E)}{d(\delta/l)} \qquad \delta/l = \sum \frac{a_n}{l}\sin\frac{n\pi}{2}$$

$$\therefore \quad W_{nn} = \frac{\pi^2}{2}n^2\left[n^2 - \frac{P}{P_E}\right] + \left(\frac{F'}{P_E}\right)$$

$$W_{nm} = (-1)^{\frac{n-1+m-1}{2}}\left(\frac{F'}{P_E}\right)$$

We write
$$\frac{\pi^2}{2}n^2\left[n^2 - \frac{P}{P_E}\right] = P(n), \qquad \frac{F'}{P_E} = S.$$

The determinants to be investigated are of the type,

$$\Delta_5 = \begin{vmatrix} P(1)+S & -S & +S & -S & +S \\ -S & P(3)+S & -S & +S & -S \\ +S & -S & P(5)+S & -S & +S \\ -S & +S & -S & P(7)+S & -S \\ +S & -S & +S & -S & P(9)+S \end{vmatrix}$$

$$\Delta_S = \begin{vmatrix} P(1)+S & -S & +S & -S & +S \\ -S & P(3)+S & -S & +S & -S \\ +S & -S & P(5)+S & -S & +S \\ -S & +S & -S & P(7)+S & -S \\ 0 & 0 & 0 & P(7) & P(9) \end{vmatrix}$$

$$= \begin{vmatrix} P(1)+S & -S & +S & -S & 0 \\ -S & P(3)+S & -S & +S & 0 \\ +S & -S & P(5)+S & -S & 0 \\ -S & +S & -S & P(7)+S & P(7) \\ 0 & 0 & 0 & P(7) & P(9) \end{vmatrix}$$

$$= \begin{vmatrix} P(1)+S & P(1) & 0 & 0 & 0 \\ P(1) & P(3)+P(1) & P(3) & 0 & 0 \\ 0 & P(3) & P(5)+P(3) & P(5) & 0 \\ 0 & 0 & P(5) & P(7)+P(5) & P(7) \\ 0 & 0 & 0 & P(7) & P(9)+P(7) \end{vmatrix}$$

$$\Delta_1 = P(1) + S$$

$$\Delta_2 = P(1)P(3) + S\left[P(1) + P(3)\right]$$

$$\Delta_3 = \Delta_2 \left[P(3) + P(5)\right] - P(3)^2 \Delta_1$$

$$= \left[P(3) + P(5)\right]\left\{P(1)P(3) + S\left[P(1) + P(3)\right]\right\} - P(3)^2\left[P(1) + S\right]$$

$$= P(1)P(3)P(5) + \left[P(3) + P(5)\right] S \left[P(1) + P(3)\right] - P(3)^2 S$$

$$= P(1)P(3)P(5) + S\left[P(1)P(3) + P(3)P(5) + P(1)P(5)\right]$$

$$\Delta_4 = \Delta_3 \cdot \left[P(5) + P(7)\right] - P(5)^2 \Delta_2$$

$$= \left[P(5) + P(7)\right]\left\{P(1)P(3)P(5) + S\left[P(1)P(3) + P(3)P(5) + P(5)P(3)\right]\right\}$$

$$- P(5)^2\left\{P(1)P(3) + S\left[P(1) + P(3)\right]\right\}$$

$$= P(1)P(3)P(5)P(7) + S\left[P(1)P(3)P(5) + P(3)P(5)P(7) + P(5)P(7)P(1) + P(7)P(1)P(3)\right]$$

— — — — — —

$$\Delta_m = \underbrace{P(1)P(3)P(5)\cdots P(2m-1)}_{m\text{-factors}} + S\left[\underbrace{P(1)P(3)\cdots P(2m-3)}_{(m-1)\text{-factors}} + \cdots\right]$$

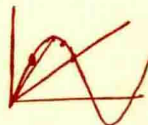
$$= P(1)P(3)P(5)\cdots P(2m-1)\left\{1 + S\sum_{n=1,3,5}^{2m-1} \frac{2}{\pi^2}\frac{1}{n^2\left[n^2 - \frac{P}{P_E}\right]}\right\}$$

When m is large

$$\Delta_{m \atop m\to\infty} \cong P(1)\cdots\cdots P(2m-1)\left[1+ \delta \sum_{n=1,3,5}^{\infty} \frac{2}{\pi^2} \frac{1}{n^2\left[n^2-\frac{P}{P_E}\right]}\right]$$

$$\cong P(1)\cdots P(2m-1)\left[1-\delta \frac{1}{\frac{P}{P_E}}\left(\frac{1}{4}-\frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)\right]$$

But for symmetrical buckling



$$\frac{\delta}{l}\,\frac{\frac{P}{P_E}}{\frac{F}{P_E}} = \left(\frac{1}{4}-\frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)$$

$$\boxed{\Delta_{m \atop m\to\infty} \cong P(1)\cdots P(2m-1)\left[1-\frac{\frac{d\left(\frac{F}{P_E}\right)}{d\xi}}{\frac{\frac{F}{P_E}}{\xi}}\right]}$$

$$\sum_{n=1,3,5}^{2m-1} \frac{2}{\pi^2}\frac{1}{\left[n^2-\frac{P}{P_E}\right]n^2} = \frac{2}{\pi^2}\frac{1}{\frac{P}{P_E}}\left[-\sum_{n=1,3,5}^{2m-1}\frac{1}{n^2}-\sum_{n=1,3,5}^{2m-1}\frac{1}{\frac{P}{P_E}-n^2}\right]$$

$$= +\frac{2}{\pi^2}\frac{1}{\frac{P}{P_E}}\left[-\sum_{n=1,3,5}^{2m-1}\frac{1}{n^2}+\frac{1}{2\sqrt{\frac{P}{P_E}}}\sum_{n=1,3,5}^{2m-1}\left\{\frac{1}{n-\sqrt{\frac{P}{P_E}}}-\frac{1}{n+\sqrt{\frac{P}{P_E}}}\right\}\right]$$

$$\sum_{n=1,3,5}^{2m-1}\frac{1}{n-\sqrt{\frac{P}{P_E}}} = \sum_{n=1,3,5}^{2m-1}\int_0^\infty e^{-x\left(n-\sqrt{\frac{P}{P_E}}\right)}dx$$

$$= \int_0^\infty e^{x\sqrt{\frac{P}{P_E}}}\sum_{n=1,3,5}^{2m-1} e^{-xn}\,dx = \int_0^\infty e^{x\sqrt{\frac{P}{P_E}}-x}\sum_{n=0,1,2}^{(m-1)} e^{-2xn}\,dx$$

$$\sum_{n=1,3,5}^{2m-1} \frac{1}{n-\sqrt{\frac{p}{p_E}}} = \int_0^\infty e^{x\sqrt{\frac{p}{p_E}}-x}\, \frac{1-e^{-2mx}}{1-e^{-2x}}\, dx$$

$$= \int_0^\infty \frac{e^{-x\left(1-\sqrt{\frac{p}{p_E}}\right)} - e^{-x\left(2m+1-\sqrt{\frac{p}{p_E}}\right)}}{1-e^{2x}}\, dx$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-t\left(\frac{1}{2}-\frac{1}{2}\sqrt{\frac{p}{p_E}}\right)} - e^{-t\left(m+\frac{1}{2}-\frac{1}{2}\sqrt{\frac{p}{p_E}}\right)}}{1-e^{-t}}\, dt$$

$$= \frac{1}{2}\left\{\psi\left[\frac{m+1-\sqrt{\frac{p}{p_E}}}{2}\right] - \psi\left[\frac{1-\sqrt{\frac{p}{p_E}}}{2}\right]\right\}$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{n^2-\frac{p}{p_E}} = \frac{1}{4\sqrt{\frac{p}{p_E}}}\left\{\psi\left(\frac{m+1-\sqrt{\frac{p}{p_E}}}{2}\right) - \psi\left(\frac{m+1+\sqrt{\frac{p}{p_E}}}{2}\right) + \psi\left(\frac{1+\sqrt{\frac{p}{p_E}}}{2}\right) - \psi\left(\frac{1-\sqrt{\frac{p}{p_E}}}{2}\right)\right\}$$

$$\psi\left(\frac{1+\sqrt{\frac{p}{p_E}}}{2}\right) = \psi\left(1-\frac{1-\sqrt{\frac{p}{p_E}}}{2}\right)$$

$$\therefore\quad \psi\left(\frac{1+\sqrt{\frac{p}{p_E}}}{2}\right) - \psi\left(\frac{1-\sqrt{\frac{p}{p_E}}}{2}\right) = \pi \cot \pi\left(\frac{1}{2}-\frac{1}{2}\sqrt{\frac{p}{p_E}}\right)$$

$$= \pi \tan \frac{\pi}{2}\sqrt{\frac{p}{p_E}}$$

$$\therefore\quad \boxed{\sum_{n=1,3,5}^{2m-1} \frac{1}{n^2-\frac{p}{p_E}} = \frac{1}{4\sqrt{\frac{p}{p_E}}}\left\{\psi\left(\frac{m+1-\sqrt{\frac{p}{p_E}}}{2}\right) - \psi\left(\frac{m+1+\sqrt{\frac{p}{p_E}}}{2}\right) + \pi \tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}\right\}}$$

$$\sum_{n=1,3,5}^{2m-1} \frac{1}{n^2} = \lim_{\sqrt{\frac{p}{p_E}} \to 0} \frac{1}{4\sqrt{\frac{p}{p_E}}} \left\{ \psi\left(\frac{m+1}{2} - \frac{1}{2}\sqrt{\frac{p}{p_E}}\right) - \psi\left(\frac{m+1}{2} + \frac{1}{2}\sqrt{\frac{p}{p_E}}\right) + \pi \tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}} \right\}$$

$$= -\frac{1}{4}\psi'\left(\frac{m+1}{2}\right) + \frac{\pi \tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}}{4\sqrt{\frac{p}{p_E}}}$$

$$\boxed{\sum_{n=1,3,5}^{\infty} \frac{1}{n^2} = -\frac{1}{4}\psi'\left(\frac{m+1}{2}\right) + \frac{\pi^2}{2}}$$

$$\therefore \frac{2}{\pi^2}\sum_{n=1,3,5}^{2m-1} \frac{1}{n^2\left[n^2 - \frac{p}{p_E}\right]} = \frac{2}{\pi^2}\frac{1}{\frac{p}{p_E}}\left[\frac{1}{4}\psi'\left(\frac{m+1}{2}\right) - \frac{\pi^2}{8} + \frac{1}{4\sqrt{\frac{p}{p_E}}}\left\{\psi\left(\frac{m+1-\sqrt{\frac{p}{p_E}}}{2}\right) - \psi\left(\frac{m+1+\sqrt{\frac{p}{p_E}}}{2}\right) + \pi\tan\frac{\pi}{2}\sqrt{\frac{p}{p_E}}\right\}\right]$$

$\psi'(1) = 1.644934$

$\psi'(1.5) = 0.934802$

$\psi'(2.0) = 0.644934$

$\psi'(2.5) = 0.490358$

$\psi'(3.0) = 0.394934$

$\psi'(3.5) = 0.330358$

$\psi'(4.0) = 0.283823$

$\psi'(4.5) = 0.248725$

$\psi'(5.0) = 0.221323$

$$\Delta_m = P(1)P(3)P(5)\cdots P(2m-1)\left\{1 + \frac{\frac{d(F/P_E)}{d\xi}}{\frac{P}{P_E}}\left[\frac{2}{\pi^2}\psi'\left(\frac{m+1}{2}\right) - \frac{1}{4} + \frac{1}{2\sqrt{\frac{P}{P_E}}}\left\{\frac{\psi\left(\frac{m+1-\sqrt{\frac{P}{P_E}}}{2}\right) - \psi\left(\frac{m+1+\sqrt{\frac{P}{P_E}}}{2}\right)}{\pi^2} + \frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\pi}\right\}\right]\right\}$$

$$\Delta_m = \underbrace{P(1)\cdots P(2m-1)}_{H}\left\{1 - \frac{\frac{d(F/P_E)}{d\xi}}{\frac{P}{P_E}}\left[\frac{1}{4} - \frac{2}{\pi^2}\psi'\left(\frac{m+1}{2}\right) - \frac{1}{2\sqrt{\frac{P}{P_E}}}\left\{\frac{\psi\left(\frac{m+1-\sqrt{\frac{P}{P_E}}}{2}\right) - \psi\left(\frac{m+1+\sqrt{\frac{P}{P_E}}}{2}\right)}{\pi^2} + \frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\pi}\right\}\right]\right\}$$

When $m \to \infty$

$$\Delta_{m}\Big|_{m\to\infty} \cong H\left\{1 - \frac{\frac{d(F/P_E)}{d\xi}}{\frac{P}{P_E}}\left[\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]\right\}$$

P/P_E	$\frac{1}{n^2[n^2-P/P_E]}$ $m=1$, $n^2=1$	$m=2$, $n^2=9$	$m=3$, $n^2=25$	$m=4$, $n^2=49$	$m=5$, $n^2=81$	$m=6$, $n^2=121$	$\frac{2}{\pi^2}\sum_{n=1,3,5}^{2m-1}\frac{1}{n^2[n^2-\frac{P}{P_E}]}$ $m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
3.61	-0.383142	0.020614	0.001870	0.000450	0.000160	0.000070	-0.0776408	-0.0734635	-0.0730846	-0.0729934	-0.0729610	-0.0729468
3.24	-0.446429	0.019290	0.001838	0.000446	0.000159	0.000070	-0.0904654	-0.0865565	-0.0861840	-0.0860936	-0.0860614	-0.0860472
2.89	-0.529101	0.018185	0.001809	0.000443	0.000158	0.000070	-0.1072183	-0.1035332	-0.1031667	-0.1030769	-0.1030449	-0.1030307
2.56	-0.641026	0.017253	0.001783	0.000439	0.000157	0.00040	-0.1298990	-0.1264029	-0.1260415	-0.1259526	-0.1259208	-0.1259066
2.25	-0.800000	0.016461	0.001758	0.000437	0.000157	0.000070	-0.1621139	-0.1587782	-0.1584220	-0.1583334	-0.1583016	-0.1582874
1.96	-1.041667	0.015783	0.001736	0.000434	0.000156	0.000069	-0.2110858	-0.2078875	-0.2075357	-0.2074478	-0.2074162	-0.2074022
1.69	-1.449275	0.015200	0.001716	0.000432	0.000156	0.00069	-0.2936846	-0.2906449	-0.2902972	-0.2902097	-0.2901780	-0.2901641
1.44	-2.272727	0.014697	0.001698	0.000429	0.000155	0.000069	-0.4605509	-0.4575727	-0.4572286	-0.4571417	-0.4571102	-0.4570963
1.21	-4.761905	0.014263	0.001681	0.000427	0.000155	0.0000069	-0.9649637	-0.9620736	-0.9617329	-0.9616464	-0.9616150	-0.9616010

$m=\infty$

-0.0729271

-0.0860275

-0.1030111

-0.1258867

-0.1582680

-0.2073837

-0.2901036

-0.4570785

-0.9615876

$$\frac{2}{\pi^2} = 0.2026424$$

For the case $P/P_E = 1$, the sign of Δ_m is same as the sign of S!

We found that all the straight positions are stable!

$$S = 13.7123\,(1 - 4.177778\xi^* + 3.133333\,\xi^{*2})$$

129

Buckled Positions

$\Delta_{\xi m} \sim = -(1 + S \sum')$ for P/P_E between 1 and 9.

P/P_E	S	$m = 1$	$m = \infty$
3.61	13.7123	+0.064634	+ 0
3.24	9.61834	−0.129873	−0.172558
2.89	6.03055	−0.353415	−0.378786
2.56	2.91381	−0.621499	−0.633190
2.25	0.252169	−0.959120	−0.960090
1.96	−1.95454	−1.412576	−1.405340
1.69	−3.684166	−2.081983	−2.068790
1.44	−4.876464	−3.245858	−3.228925
1.21	−5.38179	−6.193185	−6.175014
1.00	−4.71574	−	−
1.00	+7.23891	+	+
1.21	+14.059674	+12.567078	+12.519608

Unstable (rows 3.61 to 1.21)

$\xi^* = 1.00$, we have

$S = -0.609443$

$\frac{F/P_E}{\xi} = -0.609443$

Neutral (?)
Unstable!

Stable!!!

By inspection, the formula for stability criterion is not changed by the introduction of initial deflection:

For $\frac{a_1^0}{\pi i} = 0.100$; $\xi^* = 0.12007$, $P/P_E = 1.69$

$\xi^* = 0.31349$,

$\delta = 7.453280$,

$\delta = -0.001160$

Factor = +1.162223 (+)

Stable

Factor = −1.000511

Unstable

$P/P_E = 0.81$; $\sum^{\infty} = +1.069784$

$\frac{a_1^0}{\pi i} = 0.5$

$\xi^* = 0.22100$

$\xi^* = 0.48743$

$\delta = +3.150346$

Factor = +4.370190,

Stable

$\delta = -4.003059$

Factor = −3.28408

Unstable

Isometric Problem

$$W^* = \frac{W}{P_E l} = \frac{\pi^2}{4}\sum_{n=1,3,5}^{\infty} n^4\left(\frac{a_n}{l}\right)^2 + \frac{1}{2}\frac{P_E}{AE}\left(\frac{P}{P_E}\right)^2 + \int_0^{\xi}\left(\frac{F}{P_E}\right)(\xi)\,d\xi$$

$$\frac{\varepsilon}{l} = \frac{\pi^2}{4}\sum_{n=1,3,5}^{\infty} n^2\left(\frac{a_n}{l}\right)^2 + \frac{P_E}{AE}\frac{P}{P_E} = \text{Constant}$$

$$\frac{\partial W^*}{\partial a_n^*} = \frac{\pi^2}{2}n^4\left(\frac{a_n}{l}\right) + \frac{F}{P_E}(\xi)\sin\frac{n\pi}{2} + \frac{P_E}{AE}\frac{P}{P_E}\frac{\partial\frac{P}{P_E}}{\partial a_n^*}$$

$$\frac{\partial^2 W^*}{\partial a_n^{*2}} = \frac{\pi^2}{2}n^4 + \frac{d\frac{F}{P_E}}{d\xi}\sin^2\frac{n\pi}{2} + \frac{P_E}{AE}\left(\frac{\partial\frac{P}{P_E}}{\partial a_n}\right)^2 + \frac{P_E}{AE}\frac{P}{P_E}\frac{\partial^2\frac{P}{P_E}}{\partial a_n^{*2}} = A_{nn}$$

$$\frac{\partial^2 W^*}{\partial a_n\partial a_m} = \frac{d\frac{F}{P_E}}{d\xi}\sin\frac{n\pi}{2}\sin\frac{m\pi}{2} + \frac{P_E}{AE}\frac{\partial\frac{P}{P_E}}{\partial a_n^*}\frac{\partial\frac{P}{P_E}}{\partial a_m^*} + \frac{P_E}{AE}\frac{P}{P_E}\frac{\partial^2\frac{P}{P_E}}{\partial a_n^*\partial a_m^*} = A_{nm}$$

$$0 = \frac{\pi^2}{2}n^2\left(\frac{a_n}{l}\right) + \frac{P_E}{AE}\frac{\partial\frac{P}{P_E}}{\partial a_n^*} \qquad \frac{P_E}{AE}\frac{\partial\frac{P}{P_E}}{\partial a_n^*} = -\frac{\pi^2}{2}n^2\left(\frac{a_n}{l}\right)$$

$$0 = \frac{\pi^2}{2}n^2 + \frac{P_E}{AE}\frac{\partial^2\frac{P}{P_E}}{\partial a_n^{*2}} \qquad \frac{P_E}{AE}\frac{\partial^2\frac{P}{P_E}}{\partial a_n^{*2}} = -\frac{\pi^2}{2}n^2$$

$$0 = \frac{P_E}{AE}\frac{\partial^2\frac{P}{P_E}}{\partial a_n^*\partial a_m^*}$$

$$A_{nn} = \frac{\pi^2}{2}n^2\left[n^2 - \frac{P}{P_E}\right] + \frac{d\frac{F}{P_E}}{d\xi} + \frac{\frac{\pi^4}{4}n^4\left(\frac{a_n}{l}\right)^2}{\frac{P_E}{AE}}$$

$$A_{nm} = \frac{d\frac{F}{P_E}}{d\xi}\sin\frac{n\pi}{2}\sin\frac{m\pi}{2} + \frac{\frac{\pi^4}{4}n^2m^2\left(\frac{a_n}{l}\right)\left(\frac{a_m}{l}\right)}{\frac{P_E}{AE}}$$

$$A_{nn} = \frac{\pi^2}{2} n^2 \left[n^2 - \frac{P}{P_E} \right] + \left\{ \frac{d \frac{F}{P_E}}{d\xi} + \frac{\left(\frac{F}{P_E}\right)^2}{\frac{P_E}{AE}\left(\frac{P}{P_E} - n^2\right)^2} \right\}$$

$$A_{nm} = -(-1)^{\frac{n+m}{2}} \left\{ \frac{d \frac{F}{P_E}}{d\xi} + \frac{\left(\frac{F}{P_E}\right)^2}{\frac{P_E}{AE}\left(\frac{P}{P_E} - n^2\right)\left(\frac{P}{P_E} - m^2\right)} \right\}$$

When the column is straight, $F/P_E = 0$, the condition is same as before, and therefore it is stable even under the new point of view.

$$\frac{P_E}{AE} = \frac{EI\pi^2}{AEl^2} = \left(\frac{\pi i}{l}\right)^2$$

Section 6

Buckling of Column with One Non-linear Support, Initial Deflection and Elasticity of Machine

$$W^* = \frac{\pi^2}{4}\left[\left(\frac{a_1-a_1^0}{l}\right)^2 + \sum_{n=3,5}^{\infty} n^4\left(\frac{a_n}{l}\right)^2\right] + \frac{1}{2}\left(\frac{P_E}{AE} + \frac{kP_E}{l}\right)\left(\frac{P}{P_E}\right)^2 + \int_0^{\xi} \frac{F}{P_E}(\xi)\, d\xi$$

$$\frac{\varepsilon}{l} = \frac{\pi^2}{4}\left[\sum_{n=1,3,5}^{\infty} n^2\left(\frac{a_n}{l}\right)^2 - \left(\frac{a_1^0}{l}\right)^2\right] + \left(\frac{P_E}{AE} + \frac{kP_E}{l}\right)\left(\frac{P}{P_E}\right)$$

$$\frac{\partial W^*}{\partial a_1} = \frac{\pi^2}{2}\left(\frac{a_1-a_1^0}{l}\right) + \left(\frac{P_E}{AE} + \frac{kP_E}{l}\right)\left(\frac{P}{P_E}\right)\frac{\partial \frac{P}{P_E}}{\partial a_1} + \frac{F}{P_E}\sin\frac{\pi}{2} = 0 \qquad (1)$$

$$0 = \frac{\pi^2}{2}\left(\frac{a_1}{l}\right) + \left(\frac{P_E}{AE} + \frac{kP_E}{l}\right)\frac{\partial \frac{P}{P_E}}{\partial a_1} \qquad (2)$$

$$\frac{\partial^2 W^*}{\partial a_1^2} = A_{11} = \frac{\pi^2}{2} + \left(\frac{P_E}{AE} + \frac{kP_E}{l}\right)\left[\left(\frac{\partial \frac{P}{P_E}}{\partial a_1}\right)^2 + \frac{P}{P_E}\frac{\partial^2 \frac{P}{P_E}}{\partial a_1^2}\right] + \frac{d\frac{F}{P_E}}{d\xi}\sin^2\frac{\pi}{2} \qquad (3)$$

$$0 = \frac{\pi^2}{2} + \left(\frac{P_E}{AE} + \frac{kP_E}{l}\right)\frac{\partial^2 \frac{P}{P_E}}{\partial a_1^2} \qquad (4)$$

Substituting (2) into (1)

$$0 = \frac{\pi^2}{2}\left[\left(\frac{a_1-a_1^0}{l}\right) - \left(\frac{a_1}{l}\right)\frac{P}{P_E}\right] + \frac{F}{P_E}\sin\frac{\pi}{2}$$

$$\frac{a_1}{l} = \frac{\frac{F}{P_E}\sin\frac{\pi}{2} - \frac{\pi^2}{2}\frac{a_1^0}{l}}{\frac{\pi^2}{2}\left[\frac{P}{P_E}-1\right]} = \frac{\frac{2}{\pi^2}\frac{F}{P_E} - \frac{a_1^0}{l}}{\frac{P}{P_E}-1} \qquad (5)$$

from (2), $\left(\frac{P_E}{AE}+\frac{kP_E}{l}\right)\frac{\partial \frac{P}{P_E}}{\partial a_1} = -\frac{\pi^2}{2}\left(\frac{a_1}{l}\right) = -\frac{\frac{F}{P_E}-\frac{\pi^2}{2}\frac{a_1^0}{l}}{\frac{P}{P_E}-1}$

and by (4), $\left(\frac{P_E}{AE}+\frac{kP_E}{l}\right)\frac{\partial^2 \frac{P}{P_E}}{\partial a_1^2} = -\frac{\pi^2}{2}$

$$A_{11} = \frac{\pi^2}{2} + \frac{1}{\left(\frac{P_E}{AE}+\frac{kP_E}{l}\right)}\frac{\left(\frac{F}{P_E}-\frac{\pi^2}{2}\frac{a_1^0}{l}\right)^2}{\left(\frac{P}{P_E}-1\right)^2} - \frac{\pi^2}{2}\frac{P}{P_E} + \frac{d\frac{F}{P_E}}{d\xi}$$

$$A_{11} = \frac{\pi^2}{2}\left[1-\frac{P}{P_E}\right] + \frac{d\frac{F}{P_E}}{d\xi} + \frac{1}{\left(\frac{P_E}{AE}+\frac{kP_E}{l}\right)}\frac{\left(\frac{F}{P_E}-\frac{\pi^2}{2}\frac{a_1^0}{l}\right)^2}{\left(\frac{P}{P_E}-1\right)^2}$$

$$A_{1n} = A_{n1} = -(-1)^{\frac{1+n}{2}}\left\{\frac{d\frac{F}{P_E}}{d\xi} + \frac{1}{\left(\frac{P_E}{AE}+\frac{kP_E}{l}\right)}\frac{\left(\frac{F}{P_E}-\frac{\pi^2}{2}\frac{a_1^0}{l}\right)\frac{F}{P_E}}{\left(\frac{P}{P_E}-1\right)\left(\frac{P}{P_E}-n^2\right)}\right\}$$

$$A_{nn} = \frac{\pi^2}{2}n^2\left[n^2-\frac{P}{P_E}\right] + \left\{\frac{d\frac{F}{P_E}}{d\xi} + \frac{1}{\left(\frac{P_E}{AE}+\frac{kP_E}{l}\right)}\frac{\left(\frac{F}{P_E}\right)^2}{\left(\frac{P}{P_E}-n^2\right)^2}\right\}$$

$$A_{nm} = -(-1)^{\frac{m+n}{2}}\left\{\frac{d\frac{F}{P_E}}{d\xi} + \frac{1}{\left(\frac{P_E}{AE}+\frac{kP_E}{l}\right)}\frac{\left(\frac{F}{P_E}\right)^2}{\left(\frac{P}{P_E}-n^2\right)\left(\frac{P}{P_E}-m^2\right)}\right\}$$

Let $p_n = \frac{\pi^2}{2} n^2 \left(n^2 - \frac{P}{P_E}\right)$

we put $\left(\frac{\pi i^*}{l}\right)^2 = \frac{P_E}{AE} + \frac{kP_E}{l}$

$s = \frac{d\bar{\Gamma}_1^{-1/2}}{d\xi}$; $q_n = \left|\frac{\pi^2}{2} n^2 \left(\frac{a_n}{\pi i^*}\right)\right|$ (sign omitted)

$$q_1 = \frac{\left(\frac{F}{P_E} - \frac{\pi^2}{2}\frac{a_1^0}{l}\right)}{\frac{\pi i^*}{l}\cdot\left(\frac{P}{P_E} - 1\right)}$$

$$q_n = \frac{\frac{F}{P_E}}{\frac{\pi i^*}{l}\left(\frac{P}{P_E} - n^2\right)}$$

$$\Delta_1 = p_1 + q_1^2 + s$$

$$\Delta_2 = p_1 p_3 + p_1 q_3^2 + p_3 q_1^2 + s\left[(p_1 + p_3) + (q_1 - q_3)^2\right]$$

$$\Delta_3 = p_1 p_3 p_5 + p_1 p_3 q_5^2 + p_3 p_5 q_1^2 + p_5 p_1 q_3^2 + s\left[(p_1 p_3 + p_3 p_5 + p_5 p_1) + p_1(q_3 - q_5)^2 + p_3(q_5 - q_1)^2 + p_5(q_1 - q_3)^2\right]$$

$$\Delta_4 = p_1 p_3 p_5 p_7 + p_1 p_3 p_5 q_7^2 + p_3 p_5 p_7 q_1^2 + p_5 p_7 p_1 q_3^2 + p_7 p_1 p_3 q_5^2$$
$$+ s\left[(p_1 p_3 p_5 + p_3 p_5 p_7 + p_5 p_7 p_1 + p_7 p_1 p_3) + p_1 p_3 (q_5 - q_7)^2 + p_3 p_5 (q_7 - q_1)^2 + p_5 p_7 (q_1 - q_3)^2 + p_7 p_1 (q_3 - q_5)^2 + p_1 p_5 (q_3 - q_7)^2 + p_3 p_7 (q_1 - q_5)^2\right]$$

— — — — — — — — —

$$\Delta_1 = p_1\left[1 + \frac{g_1^2}{p_1} + s\left(\frac{1}{p_1}\right)\right]$$

$$\Delta_2 = p_1p_3\left[1 + \frac{g_1^2}{p_1} + \frac{g_3^2}{p_3} + s\left\{\frac{1}{p_1} + \frac{1}{p_3} + \frac{(g_1-g_3)^2}{p_1p_3}\right\}\right]$$

$$\Delta_3 = p_1p_3p_5\left[1 + \frac{g_1^2}{p_1} + \frac{g_3^2}{p_3} + \frac{g_5^2}{p_5} + s\left\{\frac{1}{p_1} + \frac{1}{p_3} + \frac{1}{p_5} + \frac{(g_1-g_3)^2}{p_1p_3} + \frac{(g_3-g_5)^2}{p_3p_5} + \frac{(g_5-g_1)^2}{p_5p_1}\right\}\right]$$

$$\Delta_4 = p_1p_3p_5p_7\left[1 + \frac{g_1^2}{p_1} + \frac{g_3^2}{p_3} + \frac{g_5^2}{p_5} + \frac{g_7^2}{p_7} + s\left\{\frac{1}{p_1} + \frac{1}{p_3} + \frac{1}{p_5} + \frac{1}{p_7} + \frac{(g_1-g_3)^2}{p_1p_3} + \frac{(g_3-g_5)^2}{p_3p_5} + \frac{(g_5-g_7)^2}{p_5p_7} + \frac{(g_7-g_1)^2}{p_7p_1} + \frac{(g_1-g_5)^2}{p_1p_5} + \frac{(g_3-g_7)^2}{p_3p_7}\right\}\right]$$

Assuming $a_1^\circ = 0$ and $f = \dfrac{\frac{F}{P_E}}{(\pi i^2/l)}$

$$\frac{g_1^2}{p_1} + \frac{g_3^2}{p_3} + \frac{g_5^2}{p_5} + \cdots = \frac{2f^2}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left(n^2-\frac{P}{P_E}\right)^3}$$

$$\frac{1}{p_1} + \frac{1}{p_3} + \frac{1}{p_5} + \cdots = \frac{2}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left(n^2-\frac{P}{P_E}\right)}$$

$$\frac{(g_1-g_3)^2}{p_1p_3} + \frac{(g_3-g_5)^2}{p_3p_5} + \cdots = \left(\frac{2f}{\pi^2}\right)^2\left[\left\{\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left(n^2-\frac{P}{P_E}\right)^3}\right\}\left\{\sum_{m=1,3,5}^{\infty}\frac{m^2}{\left(m^2-\frac{P}{P_E}\right)^3}\right\} - \left\{\sum_{n=1,3,5}^{\infty}\frac{1}{\left(n^2-\frac{P}{P_E}\right)^3}\right\}^2\right]$$

$$\frac{(g_n-g_m)^2}{p_np_m} = \left(\frac{2f}{\pi^2}\right)^2\frac{(n^2-m^2)^2}{n^2m^2\left(n^2-\frac{P}{P_E}\right)^3\left(m^2-\frac{P}{P_E}\right)^3} = \left(\frac{2f}{\pi^2}\right)^2\left[\frac{\left(\frac{n}{m}\right)^2 - 2 + \left(\frac{m}{n}\right)^2}{\left(n^2-\frac{P}{P_E}\right)^3\left(m^2-\frac{P}{P_E}\right)^3}\right]$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{n^2\left(n^2-\frac{P}{P_E}\right)} = -\frac{1}{\frac{P}{P_E}}\left[\frac{\pi^2}{8} - \frac{\pi}{4\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{n^2\left(n^2-\frac{P}{P_E}\right)^3} = \frac{1}{2}\frac{\partial^2}{\partial \frac{P}{P_E}^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\left(n^2-\frac{P}{P_E}\right)}$$

$$= -\frac{1}{2}\frac{\partial^2}{\partial \frac{P}{P_E}^2}\left[\frac{\pi^2}{8}\frac{1}{\frac{P}{P_E}} - \frac{\pi}{4}\frac{1}{\left(\frac{P}{P_E}\right)^{3/2}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$= -\frac{1}{2}\left[\frac{\pi^2}{4}\frac{1}{\left(\frac{P}{P_E}\right)^3} - \frac{\pi}{4}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{1}{\left(\frac{P}{P_E}\right)^{7/2}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right.$$

$$-\frac{\pi}{2}\left(-\frac{3}{2}\right)\frac{1}{\left(\frac{P}{P_E}\right)^{5/2}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\cdot\left(\frac{\pi}{2}\frac{1}{2}\frac{1}{\sqrt{\frac{P}{P_E}}}\right)$$

$$\left.-\frac{\pi}{4}\frac{1}{\left(\frac{P}{P_E}\right)^{3/2}}\left\{2\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\left(\frac{\pi^2}{16}\frac{1}{\frac{P}{P_E}}\right) + \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\cdot\left(-\frac{\pi}{8}\frac{1}{\frac{P}{P_E}^{3/2}}\right)\right\}\right]$$

$$= -\frac{1}{\left(\frac{P}{P_E}\right)^3}\left[\frac{\pi^2}{8} - \frac{15\pi}{32}\frac{1}{\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{3\pi^2}{32}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right.$$

$$\left.-\frac{\pi^3}{64}\sqrt{\frac{P}{P_E}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\cdot\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{\pi^2}{64}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{n^2\left(n^2-\frac{P}{P_E}\right)^3} = -\frac{1}{\left(\frac{P}{P_E}\right)^3}\left[\frac{\pi^2}{8} + \frac{7}{64}\pi^2\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{15\pi}{32\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right.$$

$$\left.-\frac{\pi^3\sqrt{\frac{P}{P_E}}}{64}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{\left(n^2 - \frac{P}{P_E}\right)} = \frac{\pi}{4} \frac{1}{\sqrt{\frac{P}{P_E}}} \tan \frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{\left(n^2 - \frac{P}{P_E}\right)^3} = \frac{1}{2} \frac{\partial^2}{\partial \frac{P}{P_E}^2} \sum_{n=1,3,5}^{\infty} \frac{1}{\left(\lambda^2 - \frac{P}{P_E}\right)} = \frac{1}{2} \frac{\partial^2}{\partial \frac{P}{P_E}^2} \frac{\pi}{4} \frac{1}{\sqrt{\frac{P}{P_E}}} \tan \frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$= \frac{\pi}{8}\Bigg[\frac{1}{2}\frac{3}{2} \frac{1}{\left(\frac{P}{P_E}\right)^{5/2}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + 2\left(-\frac{1}{2}\right) \frac{1}{\left(\frac{P}{P_E}\right)^{3/2}} \sec^2 \frac{\pi}{2}\sqrt{\frac{P}{P_E}} \left(\frac{\pi}{2}\frac{1}{2}\frac{1}{\sqrt{\frac{P}{P_E}}}\right)$$

$$+ \frac{1}{\sqrt{\frac{P}{P_E}}} \left\{ 2\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \left(\frac{\pi^2}{16}\frac{1}{\frac{P}{P_E}}\right) + \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \left(-\frac{\pi}{8}\frac{1}{\frac{P^{3/2}}{P_E}}\right)\right\}\Bigg]$$

$$= \frac{\pi}{8} \frac{1}{\left(\frac{P}{P_E}\right)^{5/2}} \Bigg[\frac{3}{4}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{\pi}{4}\sqrt{\frac{P}{P_E}} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{\pi^2}{8}\frac{P}{P_E} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$- \frac{\pi}{8}\sqrt{\frac{P}{P_E}} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \Bigg]$$

$$\boxed{\sum_{n=1,3,5}^{\infty} \frac{1}{\left(n^2 - \frac{P}{P_E}\right)^3} = \frac{1}{\left(\frac{P}{P_E}\right)^2}\left[\frac{3\pi}{32\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3\pi^2}{64} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{\pi^3}{64}\sqrt{\frac{P}{P_E}} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \right]}$$

$$\sum_{n=1,3,5}^{\infty} \frac{n^2}{\left(n^2 - \frac{P}{P_E}\right)^3} = \frac{P}{P_E} \sum_{n=1,3,5}^{\infty} \frac{1}{\left(n^2 - \frac{P}{P_E}\right)^3} + \sum_{n=1,3,5}^{\infty} \frac{1}{\left(n^2 - \frac{P}{P_E}\right)^2}$$

$$= \frac{1}{2}\frac{1}{\left(\frac{P}{P_E}\right)} \Bigg[\frac{3\pi}{16\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3\pi^2}{32} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{\pi^3}{32}\sqrt{\frac{P}{P_E}} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$+ \frac{\pi^2}{8} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{\pi}{4\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \Bigg]$$

$$\sum_{n=1,3,5}^{\infty} \frac{n^2}{\left(n^2-\frac{P}{P_E}\right)^3} = \frac{1}{32}\frac{\pi^2}{\frac{P}{P_E}}\left[-\frac{1}{\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + [illegible]\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + 2\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$\sum_{n=1,3,5}^{\infty} \frac{q_n^2}{p_n} = -\left(\frac{\frac{F}{P_E}}{\frac{\pi i^*}{l}}\right)^2 \frac{1}{\left(\frac{P}{P_E}\right)^3}\left[\frac{1}{4} + \frac{7}{32}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{15}{32\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{16}\left(\frac{\pi\sqrt{\frac{P}{P_E}}}{2}\right)\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$\sum_{n=1,3,5}^{\infty} \frac{1}{p_n} = -\frac{1}{\frac{P}{P_E}}\left[\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$\frac{(q_1-q_3)^2}{p_1 p_3} + \cdots = -\left(\frac{\frac{F}{P_E}}{\frac{\pi i^*}{l}}\right)^2 \frac{1}{\left(\frac{P}{P_E}\right)^4}\left[\left\{\frac{1}{4} + \frac{7}{32}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{15}{32\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{16}\left(\frac{\pi\sqrt{\frac{P}{P_E}}}{2}\right)\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}\right.$$

$$\left\{\frac{1}{16}\frac{-1}{\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{1}{16}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{1}{8}\left(\frac{\pi\sqrt{\frac{P}{P_E}}}{2}\right)\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}\frac{1}{2}$$

$$\left. + \frac{1}{4}\left\{\frac{3}{16\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3}{16}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{1}{8}\left(\frac{\pi\sqrt{\frac{P}{P_E}}}{2}\right)\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}^2\right]$$

$$\frac{(q_1-q_3)^2}{p_1 p_3} + \cdots = -\left(\frac{\frac{F}{P_E}}{\frac{\pi i^*}{l}}\right)^2 \frac{1}{\left(\frac{P}{P_E}\right)^4}\frac{1}{128}\left[\frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}\left\{3\,\frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\frac{\pi}{2}\sqrt{\frac{P}{P_E}}} - 5\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - 1\right\}\right.$$

$$\left. + \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\left\{3 + 2\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}\right]$$

$\frac{7}{32} = 0.218750$, $\frac{1}{16} = 0.062500$, $\frac{15}{32} = 0.468750$

①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫
P/P_E	$\frac{2}{\pi^2}\frac{1}{(1-\frac{P}{P_E})^3}$	$\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$	$\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$	$\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$	$\frac{7}{32}-\frac{1}{16}$③④	⑥ × ⑤	④ ÷ ③	$\frac{1}{4}-\frac{15}{32}$⑧	−[⑦+①] ÷ ①^3	⑦ + ⑨	$\frac{1}{8}$③④⑤
3.61	−0.0113975	2.984512	−0.15838	1.025084	0.248293	0.254521	−0.053067	0.274875	−0.0112528	0.529396	−0.060568
3.24	−0.0180296	2.827433	−0.32492	1.105573	0.276168	0.305324	−0.114917	0.303867	−0.0179109	0.609191	−0.126960
2.89	−0.0300155	2.670353	−0.50953	1.259621	0.303787	0.382659	−0.190810	0.339442	−0.0299161	0.722101	−0.214234
2.56	−0.0533773	2.513274	−0.72654	1.527860	0.332875	0.508586	−0.289081	0.385507	−0.0532921	0.894093	−0.348733
2.25	−0.1037529	2.356194	−1.0000	2.00000	0.366012	0.732024	−0.424443	0.448944	−0.103679	1.180968	−0.589049
1.96	−0.2290428	2.199114	−1.3764	2.894477	0.407927	1.180741	−0.625888	0.543385	−0.228982	1.724126	−1.095146
1.69	−0.6168549	2.042035	−1.9626	4.851797	0.469231	2.276614	−0.961100	0.700516	−0.616791	2.977130	−2.430568
1.44	−2.3788787	1.884955	−3.0777	10.472237	0.581334	6.087867	−1.632771	1.015361	−2.378857	7.103228	−7.594108
1.21	−21.8812655	1.727876	−6.3138	40.864070	0.900590	36.80177	−3.654082	1.962851	−21.88161	38.764621	−55.725633

134

$\frac{3}{16} = 0.1875000$ $\frac{1}{\frac{\pi i}{l}} \frac{E}{P_E} = 13.7123\, \xi^*(1 - 9.08889\,\xi^* + 1.04444\,\xi^{*2})$

①	⑬	⑭	⑮	⑯	⑰	⑱	⑲	⑳	㉑	㉒
P/P_E	$\frac{3}{16}$ ⑧ $\frac{1}{16}$ ⑨ ⑫	$\frac{3}{16}$(⑥ - ⑤) ⑫	$(P/P_E)^4$	-[⑭² - ⑪ ⑬]/⑮	$\frac{E}{P_E}/\left(\frac{\pi i}{l}\right)$	⑰²	$\alpha_1 = 1 +$ ③ ⑱	$\alpha_\infty = 1 +$ ⑩ ⑱	$\frac{3}{\pi}\frac{1}{[1-\frac{P_E}{P}]}$	$\Sigma \frac{3}{\pi}\frac{1}{[1-\frac{P_E}{P}]}$
3.61		-0.262721	169.83563		0.000000	0.000000	1.000000	1.000000	-0.0776408	-0.0776408
3.24	-0.025340	-0.355802	110.19961	-0.000147	0.880766	0.775749	0.986014	0.986106	-0.0904654	-0.090579
2.89	-0.061781	-0.486190	69.757574	-0.000208	1.468276	2.155834	0.935292	0.935506	-0.1072183	-0.107666
2.56	-0.117587	-0.689409	42.949673	-0.000319	1.804886	3.257613	0.826130	0.826375	-0.1298990	-0.130938
2.25	-0.218762	-1.043626	25.628906	-0.000544	1.923913	3.701441	0.615965	0.616238	-0.1621139	-0.164128
1.96	-0.437563	-1.755214	14.757891	-0.001069	1.852479	3.431678	0.213999	0.214208	-0.2110858	-0.214754
1.69	-1.033637	-3.570487	8.157307	-0.002599	1.612511	2.600192	-0.603941	-0.603775	-0.2936846	-0.300443
1.44	-3.418773	-9.863997	4.299817	-0.009141	1.220946	1.490709	-2.546216	-2.546183	-0.4605509	-0.474478
1.21	-26.471624	-64.072787	2.143589	-0.078405	0.687068	0.472062	-9.329314	-9.32948	-0.9649639	-1.001976
1.00	-∞	-∞	1		0	0	-∞	-∞	-∞	-∞
1.00	-∞	-∞	1		0	0	-∞	-∞	-∞	-∞
1.21	-58.508018	-64.072787	2.143589	-0.078405	1.392971	1.940368	-41.457707	-41.458376	-0.9649639	-1. 09[illegible]

Criterium $-(\alpha_1 + s$ ㉑$)$ and $-(\alpha_\infty + s$ ㉒$)$

Limit of Stability for the case $a^* \sim \to \infty$

we have $\xi = \dfrac{\frac{E}{P_E}}{\frac{P}{P_E}} \left\{ \frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \right\}$ very soft Machine!

$$\frac{\frac{d\xi}{d\left(\frac{P}{P_E}\right)}}{\frac{d\left(\frac{E}{P_E}\right)}{d\left(\frac{P}{P_E}\right)}} = \left[\left\{ \frac{\frac{d\left(\frac{E}{P_E}\right)}{d\left(\frac{P}{P_E}\right)}}{\frac{E}{P_E}} - \frac{1}{\frac{P}{P_E}} \right\} \left\{ \frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \right\} \right.$$

$$\left. + \left\{ \frac{1}{4\pi\left(\frac{P}{P_E}\right)^{3/2}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \left(\frac{\pi}{4} \frac{1}{\sqrt{\frac{P}{P_E}}} \right) \right\} \right] \frac{\frac{E}{P_E}}{\frac{P}{P_E}} \frac{1}{\frac{d\left(\frac{E}{P_E}\right)}{d\left(\frac{P}{P_E}\right)}} = \frac{\xi}{\frac{E}{P_E}}$$

$$= \frac{1}{\frac{P}{P_E}} \left\{ \frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \right\}$$

~~$\frac{\frac{d\left(\frac{E}{P_E}\right)}{d\left(\frac{P}{P_E}\right)}}{\frac{E}{P_E}} \left\{ \frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \right\}$~~ $= \left\{ \frac{\frac{d\left(\frac{E}{P_E}\right)}{d\left(\frac{P}{P_E}\right)}}{\frac{E}{P_E}} - \frac{1}{\frac{P}{P_E}} \right\} \left\{ \frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \right\}$

$$+ \frac{1}{\frac{P}{P_E}} \left\{ \frac{1}{4\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{1}{8} \sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \right\}$$

$$\frac{1}{4} - \frac{1}{2\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} = -\frac{1}{8} - \frac{1}{8}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} + \frac{1}{4\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

$$\boxed{\frac{3}{8} + \frac{1}{8}\tan^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - \frac{3}{4\pi\sqrt{\frac{P}{P_E}}} \tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}} = 0}$$

Condition of $\quad 1 - \dfrac{\frac{d\left(\frac{E}{P_E}\right)}{d\xi}}{\frac{\frac{E}{P_E}}{\xi}} = 0$

for $\frac{\pi}{2}\sqrt{\frac{P}{P_E}} \leqq \frac{3\pi}{2}$

✓ this condition cannot be satisfied, except at $\frac{P}{P_E} = 0$ which is trivial. Therefore the only other possibility is $\dfrac{d\left(\frac{E}{P_E}\right)}{d\left(\frac{P}{P_E}\right)} \to \infty$, which gives the conclusion that the limit of stability occurs when $\dfrac{d\frac{P}{P_E}}{d(\xi/\xi)} = 0$

At the limit of stability, we have

$$1-\left(\frac{\frac{E}{P_E}}{\frac{\pi i^*}{l}}\right)^2\frac{1}{\left(\frac{P}{P_E}\right)^3}\left[\frac{1}{4}+\frac{7}{32}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}-\frac{15}{32\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}-\frac{1}{16}\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right]$$

$$-\frac{1}{\left(\frac{P}{P_E}\right)}\frac{d\left(\frac{E}{P_E}\right)}{d\xi}\left[\left\{\frac{1}{4}-\frac{1}{4\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}+\left(\frac{\frac{E}{P_E}}{\frac{\pi i^*}{l}}\right)^2\frac{1}{128\left(\frac{P}{P_E}\right)^3}\left\{\frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}\left(3\frac{\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}-5\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}-1\right)\right.\right.$$

$$\left.\left.+\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\left(3+2\,\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\,\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)\right\}\right]=0$$

From the equilibrium condition,

$$\frac{1}{\left(\frac{P}{P_E}\right)}\frac{d\frac{E}{P_E}}{d\xi}=\frac{\dfrac{\frac{d\left(\frac{E}{P_E}\right)}{d\left(P/P_E\right)}}{\left(\frac{E}{P_E}\right)}}{\left\{\dfrac{\frac{d\left(\frac{E}{P_E}\right)}{d\left(\frac{P}{P_E}\right)}}{\frac{E}{P_E}}-\dfrac{1}{\frac{P}{P_E}}\left\{\left\{\frac{1}{4}-\frac{1}{4\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right)}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}+\frac{1}{8}\left\{\frac{1}{\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_E}}-\sec^2\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right\}\frac{1}{\frac{P}{P_E}}\right.\right.}$$

Substituting into and multiply by $\left(\frac{\pi i^*}{l}\right)^2 \frac{P}{P_E}$, we get

$$\frac{1}{8}\left(\frac{\pi i^*}{l}\right)^2\left(3\frac{\tan\theta}{\theta}-2-\sec^2\theta\right)-\left(\frac{F}{P}\right)^2\left[\chi\cdot\left(\frac{1}{4}-\frac{\tan\theta}{4\theta}\right)+\frac{1}{8\frac{P}{P_E}}\left(\frac{\tan\theta}{\theta}-\sec^2\theta\right)\right]\left[\frac{1}{4}+\frac{7}{32}\sec^2\theta-\frac{15}{32\theta}\tan\theta-\frac{1}{16}\theta\tan\theta\sec^2\theta\right]$$

$$-\frac{1}{128}\left(\frac{F}{P}\right)^2\frac{1}{\frac{P}{P_E}}\left\{\frac{\tan\theta}{\theta}\left(\frac{\tan\theta}{\theta}-5\sec^2\theta-1\right)+\sec^2\theta(3+2\theta\tan\theta)\right\}$$

$$-\chi\frac{1}{128}\left(\frac{F}{P}\right)^2\left\{\frac{\tan\theta}{\theta}\left(\frac{\tan\theta}{\theta}-5\sec^2\theta-1\right)+\sec^2\theta(3+2\theta\tan\theta)\right\}=0 \quad \text{where} \quad \chi=\left(\frac{\frac{d\frac{F}{P_E}}{d\,P/P_E}}{\frac{F}{P_E}}-\frac{1}{\frac{P}{P_E}}\right)$$

$$\nearrow A$$

$$0=\left(\frac{\pi i^*}{l}\right)^2\left(2+\sec^2\theta-3\frac{\tan\theta}{\theta}\right)+\left(\frac{F}{P}\right)^2\left\{\chi\left[\left(\frac{1}{4}-\frac{\tan\theta}{4\theta}\right)\left(2+\frac{7}{4}\sec^2\theta-\frac{15}{4\theta}\tan\theta-\frac{1}{2}\theta\tan\theta\sec^2\theta\right)\right.\right.$$

$$\left.+\frac{1}{16}\frac{\tan\theta}{\theta}\left(3\frac{\tan\theta}{\theta}-5\sec^2\theta-1\right)+\frac{1}{16}\sec^2\theta(3+2\theta\tan\theta)\right]$$

$$+\frac{1}{\frac{P}{P_E}}\left[\left(\frac{\tan\theta}{\theta}-\sec^2\theta\right)\left(\frac{1}{4}+\frac{1}{32}\sec^2\theta-\frac{15}{32\theta}\tan\theta-\frac{1}{16}\theta\tan\theta\sec^2\theta\right)+\frac{1}{16}\frac{\tan\theta}{\theta}\left(3\frac{\tan\theta}{\theta}-5\sec^2\theta-1\right)+\frac{1}{16}\sec^2\theta(3+2\theta\tan\theta)\right]$$

However $A = \frac{1}{2} + \frac{7}{16}\sec^2\theta - \frac{15}{16}\frac{\tan\theta}{\theta} - \frac{1}{8}\theta\tan\theta\sec^2\theta - \frac{1}{2}\frac{\tan\theta}{\theta} - \frac{1}{16}\frac{\tan\theta}{\theta}\sec^2\theta + \frac{15}{16}\frac{\tan^2\theta}{\theta^2} + \frac{1}{8}\tan^2\theta\sec^2\theta$

$+ \frac{3}{16}\frac{\tan^2\theta}{\theta^2} - \frac{5}{16}\frac{\tan\theta}{\theta}\sec^2\theta - \frac{1}{16}\frac{\tan\theta}{\theta} + \frac{3}{16}\sec^2\theta + \frac{1}{8}\theta\tan\theta\sec^2\theta$

$= \frac{1}{2} + \frac{5}{8}\sec^2\theta - \frac{3}{2}\frac{\tan\theta}{\theta} - \frac{12}{16}\frac{\tan\theta}{\theta}\sec^2\theta + \frac{9}{8}\frac{\tan^2\theta}{\theta^2} + \frac{1}{8}\tan^2\theta\sec^2\theta$

$= \frac{9}{8} + \frac{6}{8}\tan^2\theta - \frac{9}{4}\frac{\tan\theta}{\theta} - \frac{3}{4}\frac{\tan^3\theta}{\theta} + \frac{9}{8}\frac{\tan^2\theta}{\theta^2} + \frac{1}{8}\tan^4\theta$ $\qquad \frac{24}{12}\ \frac{36}{16}$

$= \left[\frac{3}{8} + \frac{1}{8}\tan^2\theta - \frac{3}{8}\frac{\tan\theta}{\theta}\right]\left[3 + \tan^2\theta - 3\frac{\tan\theta}{\theta}\right]$ O.K.

Similarly $B = \left[3 + \tan^2\theta - 3\frac{\tan\theta}{\theta}\right]\left[\frac{1}{16}\theta\tan\theta\sec^2\theta + \frac{3}{32}\frac{\tan\theta}{\theta} - \frac{3}{32}\sec^2\theta\right]$

$0 = \left(\frac{\pi i^*}{l}\right)^2 + \left(\frac{F}{P}\right)^2\left\{\chi\left(\frac{3}{8} + \frac{1}{8}\tan^2\theta - \frac{3}{8}\frac{\tan\theta}{\theta}\right) + \frac{1}{\frac{P}{P_E}}\left[\frac{1}{16}\theta\tan\theta\sec^2\theta + \frac{3}{32}\frac{\tan\theta}{\theta} - \frac{3}{32}\sec^2\theta\right]\right\}$

$= \frac{d\left(\frac{\varepsilon}{l}\right)}{d\left(\frac{P}{P_E}\right)}$ $\quad \therefore$ Proved !!!

139

$$\prod_{n=1,3,5}^{n=2m-1} \frac{\pi^2}{2} n^2 \left[n^2 - \frac{P}{P_E} \right] =$$

$\frac{\gamma-1}{\gamma}$ $p\frac{dy}{dx}$ $p\frac{dy}{dx}$

$p\frac{dy}{dx}=0$ $p\frac{dy}{dx}=0$

150 153 153

γ γ γ $\gamma - 1$ $\frac{\gamma-1}{\gamma}$

$\dot{\gamma}$

$\int \int \int$ $n^2 - n^2 \int \frac{\pi^2}{2} n^2 \left[n^2 - \frac{P}{P_E} \right]$

$\int$ S S S S

$\frac{dy}{dx}$

P(1) + P(2) + P(3) + P(4) + P(5)

[General Information]
书名=钱学森力学手稿　　7
作者=钱学森著
页数=144
SS号=13387435
出版日期=2012.06